Sparse Multidimensional Representation using Shearlets

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ABSTRACT

In this paper we describe a new class of multidimensional representation systems, called shearlets. They are obtained by applying the actions of dilation, shear transformation and translation to a fixed function, and exhibit the geometric and mathematical properties, e.g., directionality, elongated shapes, scales, oscillations, recently advocated by many authors for sparse image processing applications. These systems can be studied within the framework of a generalized multiresolution analysis. This approach leads to a recursive algorithm for the implementation of these systems, that generalizes the classical cascade algorithm.

Keywords: Affine systems, curvelets, geometric image processing, shearlets, sparse representation, wavelets

1. INTRODUCTION

The importance of wavelets in signal processing applications is widely acknowledged. Indeed, they provide optimal approximation, in a certain sense, for one dimensional piecewise continuous functions.[1,2] On the other hand, it is also well-known that wavelets do not perform as well in dimensions larger than one. This situation is illustrated, for example, by the classical problem of representing a natural image using a 2–D wavelet basis. Natural images exhibit edges, that is, discontinuities along curves. Because these discontinuities are spatially distributed, they interact extensively with the elements of the wavelet basis, and so the wavelet representation is not sparse, that is, "many" wavelet coefficients are needed to accurately represent the edges.

This limitation has led to several new constructions, in order to handle efficiently the geometrical features of multidimensional signals. These constructions include the *directional wavelets*,[3] the *complex wavelets*,[4] the *ridgelets*,[5] the *curvelets*[6,7] and the *contourlets*.[8] The main idea, in all of these constructions, is that, in order to obtain efficient representations of multivariable functions with spatially distributed discontinuities, such representations must contain basis elements with many more shapes and directions than the classical wavelet bases. One of the most successful construction based on this idea are the curvelets of Candès and Donoho, that achieve an (almost) optimal approximation property for 2–D piecewise smooth functions with discontinuities along C^2 curves.[7]

In this paper we show that it is possible to obtain efficient representations of multivariable functions using affine systems of the form

$$\mathcal{A}_{AB}(\psi) = \{\psi_{i,j,k} = |\det A|^{i/2}\, \psi(B^j\, A^i x - k) : \, i, j \in \mathbb{Z}, k \in \mathbb{Z}^2\}, \tag{1}$$

where $A = \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

These systems are a special case of a new class of analyzing functions called **affine systems with composite dilations**. One advantage of this approach is that these systems can be studied within the framework

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of a generalized Multi-Resolution Analysis, and this is very relevant for the discrete implementation of these representations in terms of multidimensional filter banks.

The paper is organized as follows. In Section 2 we introduce the continuous shearlet transform and show its connection with the discrete systems $\mathcal{A}_{AB}(\psi)$, given by (1). In Section 3 we describe a general framework for the study of the affine systems $\mathcal{A}_{AB}(\psi)$, based on a generalized Multi-Resolution Analysis (MRA). In particular, we deduce an appropriate scaling equation associated with this MRA and a recursive algorithm for the computation of the coefficients associated with these transformations.

2. CONTINUOUS SHEARLETS

Let

$$M_{as} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} = \begin{pmatrix} a & \sqrt{a}\, s \\ 0 & \sqrt{a} \end{pmatrix}, \tag{2}$$

where $(a, s) \in \mathbb{R}^+ \times \mathbb{R}$, and consider the affine systems

$$\mathcal{A}_{M_{as}}(\psi) = \mathcal{A}_{ast}(\psi) = \{\psi_{ast}(x) = a^{-3/4}\, \psi(M_{as}^{-1}(x-t)) : \, a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2\}. \tag{3}$$

Observe that the matrix M_{as} is the composition of the non-isotropic dilation $\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$, and the **shearing transformation** $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. We will be interested in the affine systems $\mathcal{A}_{ast}(\psi)$ generated by functions ψ for which

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\, \hat{\psi}_2(\frac{\xi_2}{\xi_1}), \tag{4}$$

where ψ_1 is a continuous wavelet, for which $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{\psi}_1 \subset [-2, 1/2] \cup [1/2, 2]$, and ψ_2 is chosen such that $\hat{\psi}_2 \in C^\infty(\mathbb{R})$, $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$, with $\hat{\psi}_2 > 0$ on (-1,1), and $\|\psi_2\| = 1$. There are several examples of functions ψ_1, ψ_2 satisfying these properties.[9]

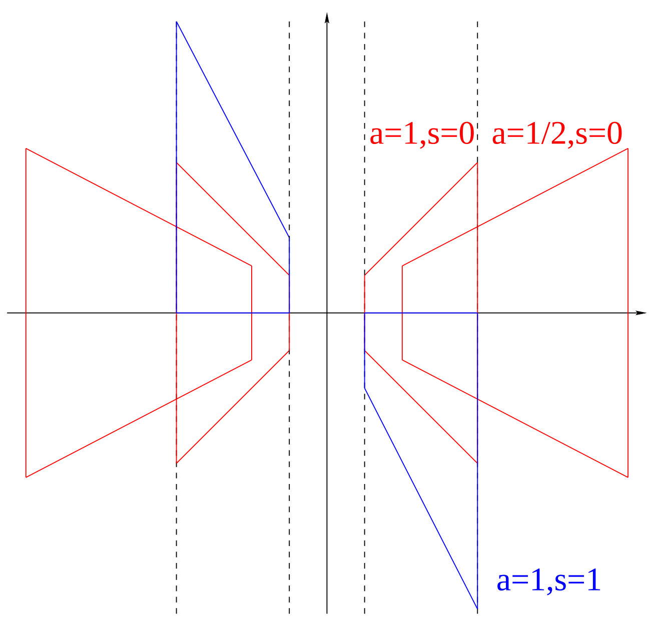

Figure 1. Support of the shearlets $\hat{\psi}_{ast}$ (in the frequency domain) for different values of a and s.

Under these assumptions on ψ, it is not hard to show that the family $\{\psi_{ast}(x) : a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2\}$ is a reproducing system for $L^2(\mathbb{R}^2)$, that is, it satisfies the Calderòn formula

$$\|f\|^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_0^\infty |\langle f, \psi_{ast}\rangle|^2 \frac{da}{a^3}\, ds\, dt,$$

for all $f \in L^2(\mathbb{R}^2)$ (details can be found in Kutyniok and Labate[10]). We will use the terminology of (continuous) **shearlets** to denote these collections of reproducing functions, and we define the **continuous shearlet transform** as the function

$$S_f(a, s, t) = \langle f, \psi_{ast}\rangle, \quad a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2.$$

The geometrical properties of the shearlets are more evident in the frequency domain. Since

$$\hat{\psi}_{ast}(\xi) = a^{\frac{3}{4}}\, e^{-2\pi i \xi t}\, \hat{\psi}_1(a\,\xi_1)\, \hat{\psi}_2(a^{-1/2}(s + \frac{\xi_2}{\xi_1})),$$

it is clear that the function $\hat{\psi}_{ast}$ has frequency support

$$\operatorname{supp} \hat{\psi}_{ast} \subset \{(\xi_1, \xi_2) : \xi_1 \in [-2/a, -1/(2a)) \cup [1/(2a), 2/a],\ |s + \xi_2/\xi_1| \le \sqrt{a}\}.$$

Thus, the shearlets are oriented waveforms, with orientation controlled by the shear parameter s, and they become increasingly thin at fine scales (for $a \to 0$). Figure 1 shows the support of the shearlets in the frequency domain for some values of a and s.

It is important to emphasize the special role of the matrices $\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ and $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ in this construction. The first matrix controls the 'scale' of the shearlets, by applying a different dilation factor along the two axes. This ensures that the frequency support of the shearlets becomes increasingly elongated at finer scales. The shear matrix $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, on the other hand, is not expansive, and only controls the orientation of the shearlets.

These geometrical features are similar, for some aspects, to the recently introduced continuous curvelet transform of Candès and Donoho.[11] The continuous curvelet transform is defined as $\Gamma_f(a, \theta, t) = \langle f, \gamma_{a\theta t}\rangle$, where $\gamma_{a\theta t}$ is obtained by applying translations by t and rotations by θ to appropriate functions γ_a, $a \in \mathbb{R}^+$, where a is a scale parameter. Observe that, unlike the shearlets, the curvelets are not generated by a simple affine transformation of a single function γ.

2.1. Resolution of edges using the continuous shearlets

Consider a 2–D function f which is smooth away from a discontinuity along a curve. This is a reasonable model for the situation one typically encounters in image processing.

It is known[12] that, if ψ is a 'nice' continuous wavelet, then the continuous wavelet transform $\mathcal{W}_f(a, t) = \langle f, \psi_{at}\rangle$, where $\psi_{at}(x) = a^{-1}\psi(a^{-1}(x - t))$, is able to localize the singularities of f in the following sense. For $a \to 0$, the function $\mathcal{W}_f(a, t)$ tends rapidly to zero, when t is outside the singularity, and $\mathcal{W}_f(a, t)$ tends to zero slowly when t is on the singularity.

The continuous shearlets are not only able to locate a discontinuity curve, but also to identify its orientation. That is, for $a \to 0$, the shearlet transform $S_f(a, s, t)$ tends rapidly to zero unless t is at the singularity *and* s describes the direction that is perpendicular to the discontinuity curve. The following example is a special case of this general property[10]:

Example 2.1. *Let $f = \chi_D$, where D is the unit disc in $\mathbb{R}^2$, then, for $a \to 0$,*

- *if $t \in \partial D$ and s describes the direction normal to ∂D, then $|S_f(a, s, t)| \le C\, a^{3/4}$;*
- *otherwise, for each $N = 1, 2, \ldots$, $|S_f(a, s, t)| \le C\, a^N$.*

Observe that the same property holds for the continuous curvelet transform of Candès and Donoho.[11]

2.2. Discretization of the continuous shearlet transform and discrete shearlets

By sampling the continuous shearlet transform $S_f(a,s,t) = \langle f, \psi_{ast}\rangle$ on an appropriate discrete set, it is possible to obtain a frame or even a tight frame for $L^2(\mathbb{R}^2)$. It is reasonable to expect that the resulting discrete systems will inherit some basic geometric properties of the corresponding continuous systems and, thus, their ability to 'localize' spatially distributed discontinuities.

In order to discretize, let us replace the continuous matrices M_{as}, given by (2), with the discrete set

$$M_{i,j} = \begin{pmatrix} 1 & j\,2^{i/2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^i & 0 \\ 0 & 2^{i/2} \end{pmatrix} = \begin{pmatrix} 2^i & 0 \\ 0 & 2^{i/2} \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = B^j A^i, \tag{5}$$

where $i, j \in \mathbb{Z}$, and A and B are the matrices defined after equation (1). Also, let the continuous translation variable $t \in \mathbb{R}^2$ be replaced by a discrete lattice. Then the affine system (3) gives us the discrete system (1). Observe that this discretization procedure is similar to the one that relates the continuous curvelet transform to the (discrete) curvelets.[13]

As a special case of systems of the form (1), we will construct a 'discretized' version of the continuous shearlets. For any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_1 \neq 0$, define $\psi \in L^2(\mathbb{R}^2)$ by

$$\hat{\psi}(\xi) = \hat{\psi}_1(4\,\xi_1)\,\hat{\psi}_2\Big(\frac{\xi_2}{\xi_1}\Big). \tag{6}$$

Let $\psi_1 \in L^2(\mathbb{R})$ be a one-dimensional dyadic wavelet with $\operatorname{supp}\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, and $\psi_2 \in L^2(\mathbb{R})$ be another band-limited function with $\operatorname{supp}\hat{\psi}_2 \subset [-1, 1]$ and satisfying

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}_2(\omega + j)|^2 = 1 \quad \text{a.e. } \omega \in \mathbb{R}. \tag{7}$$

Recall that, since ψ_1 is a dyadic wavelet, it satisfies the Calderòn equation:

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}_1(2^j\omega)|^2 = 1 \quad \text{a.e. } \omega \in \mathbb{R}. \tag{8}$$

There are several choices of functions ψ_1 and ψ_2 satisfying these properties. We will choose ψ_1 to be the Lemariè-Meyer wavelet and ψ_2 to be an arbitrary C^∞ bump function. It follows that $\hat{\psi}$, given by (6), is in $C^\infty(\mathbb{R}^2)$ and this implies that $|\psi(x)| \le K_N\,(1+|x|)^{-N}$, $K_N > 0$, for any $N \in \mathbb{N}$, and, thus, the function ψ is *well-localized*.

Using (7) and (8) it is easy to see that

$$\begin{aligned} \sum_{i,j\in\mathbb{Z}} |\hat{\psi}((B^T)^j\,A^i\,\xi)|^2 &= \sum_{ij\in\mathbb{Z}} |\hat{\psi}_1(2^{s+i}\,\xi_1)|^2\,|\hat{\psi}_2(2^{-i/2}\,\frac{\xi_2}{\xi_1} + j)|^2 \\ &= \sum_{i\in\mathbb{Z}} |\hat{\psi}_1(2^{s+i}\,\xi_1)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}_2(2^{-i/2}\,\frac{\xi_2}{\xi_1} + j)|^2 = 1 \text{ a.e.} \end{aligned}$$

This observation implies that, for this choice of ψ, the system $\{\psi_{ijk} : i, j \in \mathbb{Z}, k \in \mathbb{Z}^2\}$, given by (1), is a tight frame for $L^2(\mathbb{R}^2)$, that is,

$$\sum_{i,j,k} |\langle f, \psi_{i,j,k}\rangle|^2 = \|f\|^2, \text{ for all } f \in L^2(\mathbb{R}^2).$$

Thus, the functions ψ_{ijk} are a tight frame of well-localized oscillatory waveforms, with many directions depending on j, and needle-like for $i \to \infty$. We refer to Guo at al.[9] for details about this construction. We use the terminology of **discrete shearlets** or simply **shearlets** to refer to these systems.

3. WAVELETS WITH COMPOSITE DILATIONS

The authors and their collaborators have developed a general framework for the study of shearlets and more general systems[9, 14] . The **affine systems with composite dilations** are the collections of the form

$$\mathcal{A}_{AB}(\Psi) = \{D_{A^i}\, D_{B_j}\, T_k\, \Psi : \; k \in \mathbb{Z}^2, i, j \in \mathbb{Z}, \}, \tag{9}$$

where $\Psi = (\psi^1, \dots, \psi^L) \subset L^2(\mathbb{R}^2)$, T_k are the **translations**, defined by $T_k\, f(x) = f(x-k)$, D_A are the **dilations**, defined by $D_A\, f(x) = |\det A|^{-1/2}\, f(A^{-1}x)$, and A, $\{B_j : j \in \mathbb{Z}\}$, are invertible 2×2 matrices. By choosing Ψ, A, and B_j appropriately, one can make $\mathcal{A}_{AB}(\Psi)$ an orthonormal (ON) basis or, more generally, a tight frame for $L^2(\mathbb{R}^2)$. In this case, we call Ψ an AB–**wavelet**. It is clear that the (discrete) shearlets that we constructed in the last section are a special case of AB–wavelets, where $\{B_j = B^j : j \in \mathbb{Z}\}$ and B, A are the matrices defined after equation (1). Many other such wavelets can be constructed by choosing A to be an expanding matrix, and $\{B_j : j \in \mathbb{Z}\}$ to be a collection of non-expanding matrices of some special form, including, for example, the case where $\{B_j : j \in \mathbb{Z}\}$ is a finite group of matrices.[15] Generalizations to higher dimensions are also possible, but will not be addressed in this paper.

3.1. The theory of AB Multiresolution Analysis

Associated with the affine systems with composite dilations is the following generalization of the classical Multiresolution Analysis.

Let $\{B_j : j \in \mathbb{Z}\}$ be a collection of invertible 2×2 matrices with $|\det B_j| = 1$ and A be an invertible 2×2 matrix with integer entries. We say that a sequence $\{V_i\}_{i\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^2)$ is an AB **Multiresolution Analysis (**AB**–MRA)** if the following holds:

(i) $D_{B_j}\, T_k\, V_0 = V_0$, *for any* $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2$;

(ii) *for each* $i \in \mathbb{Z}$, $V_i \subset V_{i+1}$, *where* $V_i = D_a^{-i}\, V_0$;

(iii) $\bigcap V_i = \{0\}$ *and* $\overline{\bigcup V_i} = L^2(\mathbb{R}^2)$;

(iv) *there exists* $\phi \in L^2(\mathbb{R}^2)$ *such that* $\Phi_B = \{D_{B_j}\, T_k\, \phi : j \in \mathbb{Z},\, k \in \mathbb{Z}^2\}$ *is a tight frame for* V_0.

The space V_0 is called an AB **scaling space** and the function ϕ is an AB **scaling function** for V_0. If, in addition, Φ_B is an orthonormal basis, then we say that ϕ is an **ON** AB **scaling function**.

It is clear from this definition that, unlike the classical MRA, the scaling space is not only invariant with respect to the integer translations, but also to the B_j dilations. As in the classical MRA, the following fact is easy to verify.

THEOREM 3.1. *Let* $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^2)$ *be such that* $\{D_{B_j}\, T_k\, \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1, \dots, L\}$ *is an orthonormal basis (resp. tight frame) for* W_0*, where* W_0 *is the orthogonal complement of* V_0 *in* V_1*, that is,* $W_0 = V_1 \cap (V_0)^\perp$*. Then* Ψ *is an orthonormal (resp. tight frame) AB–multiwavelet.*

We will now apply the framework of the AB–MRA we have just introduced to construct new examples of AB–wavelets for $L^2(\mathbb{R}^2)$. For simplicity, we will consider a wavelet ψ of 'Shannon type', that is, the Fourier transform of the wavelet is the characteristic function of a set: $\hat{\psi} = \chi_I$, $I \subset \mathbb{R}^2$. However, the AB wavelets need not be of this form in general.

EXAMPLE 3.2. This construction is illustrated in Figure 2.

It is convenient to work in the frequency domain, that we will denote by $\widehat{\mathbb{R}}^2$. Let $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ and $B_j = B^j$, $j \in \mathbb{Z}$, where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $S_0 = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : |\xi_1| < \frac{1}{4}\}$. This is the vertical strip of width $\frac{1}{2}$ bounded by the lines $\pm\frac{1}{4}$ (see Figure 2). Then $S_i = A^i\, S_0$, $i \in \mathbb{Z}$, are the vertical strips $\{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : |\xi_1| < 2^{i-2}\}$. Observe that

$$(B^T)^j\, \xi = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ j\xi + \xi_2 \end{pmatrix},$$

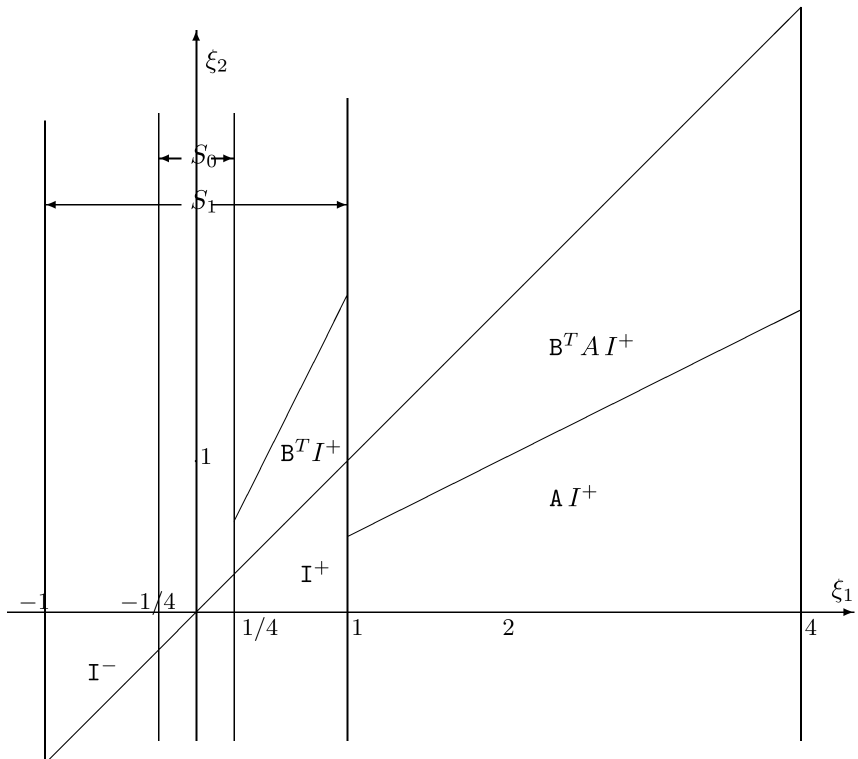

Figure 2. Example of AB wavelet. The figure shows the action of the matrices A and B on the trapezoid I^+.

and, thus, $(B^T)^j\, S_0 \subseteq S_0$, for each $j \in \mathbb{Z}$. In addition, we clearly have that (i) $S_i \subset S_{i+1}$, (ii) $\bigcup_{i\in\mathbb{Z}} S_i = \widehat{\mathbb{R}}^2$, (iii) $\bigcap_{i\in\mathbb{Z}} S_i = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : \xi_1 = 0\}$. For $S \subset \widehat{\mathbb{R}}^2$, we use the notation $L^2(S) = \{f \in L^2(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset S\}$. From the observations that we made about the sets S_i, it follows that:

(i) $D^j_{B^T}\, T_k\, L^2(S_0) = L^2(S_0)$, for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2$,

(ii) $L^2(S_i) \subset L^2(S_{i+1})$,

(iii) $\bigcap_{i\in\mathbb{Z}} L^2(S_i) = \{0\}$ and $\overline{\bigcup_{i\in\mathbb{Z}} L^2(S_i)} = L^2(\widehat{\mathbb{R}}^2)$.

Finally, let ϕ be given by $\hat{\phi} = \chi_U$, where $U = U^+ \cup U^-$, and U^+ is the triangle of vertices $(0,0)$, $(\frac{1}{4}, 0)$, , $(\frac{1}{4}, \frac{1}{4})$ and $U^- = \{\xi \in \widehat{\mathbb{R}}^2 : -\xi \in U^+\}$. Then it is simple to show that $S_0 = \cup_{j\in\mathbb{Z}}(B^T)^j U$, where the union is disjoint, and, thus, $\Phi_B = \{D_{B^j}\, T_k\, \phi : j \in \mathbb{Z},\, k \in \mathbb{Z}^2\}$ is a tight frame for V_0. In addition, Φ_B is semi-orthogonal, that is, $D_{B_{j_1}}\, T_k\, \phi \perp D_{B_{j_2}}\, T_{k'}\, \phi$ for any $j_1 \neq j_2$, $j_1, j_2 \in \mathbb{Z}$, $k, k' \in \mathbb{Z}^2$. Thus, the sequence $\{L^2(S_i) = V_i : i \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^2)$ is an AB–MRA.

In order to construct an AB–wavelet, let $R_0 = S_1 \backslash S_0$. Then $W_0 = L^2(R_0)$ is the orthogonal complement of V_0 in V_1. Next, consider the set $I = I^+ \cup I^-$, contained in R_0, where: I^+ is the trapezoid with vertices $(\frac{1}{4}, 0)$, $(\frac{1}{4}, \frac{1}{4})$, $(1, 0)$, $(1, 1)$, and $I^- = -I^+$ (see Figure 2). Then an observation similar to the one we made before shows that $\{D_{B_j}\, T_k\, \psi : j \in \mathbb{Z},\, k \in \mathbb{Z}^2$, where $\hat{\psi} = \chi_I$, is a tight frame for W_0, and, thus, by Theorem 3.1, ψ is a tight frame AB–wavelet.

3.2. A cascade algorithm for AB wavelets

As in the classical MRA, the AB scaling function ϕ determines the AB–MRA completely. Since $\phi \in V_1$, then $\phi(A^{-1}x) \in V_0$, and so

$$\phi(A^{-1}x) = \sum_{k\in\mathbb{Z}^2} \sum_{j\in\mathbb{Z}} a_{kj}\, \phi(B_j\, x - k),$$

or, equivalently,

$$\phi(x) = \sum_{k\in\mathbb{Z}^2}\sum_{j\in\mathbb{Z}} a_{kj}\,\phi(B_j\,A\,x-k).$$

Thus:

$$\hat{\phi}(\xi) = \sum_{j\in\mathbb{Z}} m_j((B_j^T)^{-1}\,A^{-1}\xi)\,\hat{\phi}((B_j^T)^{-1}\,A^{-1}\,\xi), \tag{10}$$

where $m_j(\xi) = \sum_{k\in\mathbb{Z}^2} a_{kj}\,e^{-2\pi ik\xi}$. Equation (10) is the **scaling equation** associated with the AB–MRA. Observe that this equation involves countably many 'filters' $m_j(\xi)$, as compared to the scaling equation associated with the classical MRA, that involves only one filter $m(\xi)$.

In the following, we will examine the special case where $B_j = B^j$, $j\in\mathbb{Z}$, and A, B are chosen as in Example 3.2. In addition, we assume that $m_j(\xi)\equiv 0$ for $j\neq 0,-1$, and, thus, the scaling equation associated with the AB–MRA has the form

$$\hat{\phi}(\xi) = m_0(A^{-1}\xi)\,\hat{\phi}(A^{-1}\,\xi) + m_1(B^T\,A^{-1}\xi)\,\hat{\phi}(B^T\,A^{-1}\,\xi). \tag{11}$$

Let us observe that the AB scaling equation associated with Example 3.2 is exactly of this form. However, we need not assume that $\hat{\phi}$ is the characteristic function of a set, in general.

We have the result[16]:

Theorem 3.3. *For a given function $\phi\in L^2(\mathbb{R}^2)$, let*

$$\hat{\psi^\ell}(\xi) = m_0^\ell(A^{-1}\xi)\hat{\phi}(A^{-1}\xi) + m_1^\ell(B^TA^{-1}\xi)\hat{\phi}(B^TA^{-1}\xi) \quad \textit{where } \ell = 0,\ldots,L, \tag{12}$$

and $\phi=\psi^0$. If

$$\sum_{\ell=0}^{L} m_k^\ell((B^T)^k(\xi+\alpha_i))\overline{m_{k'}^\ell((B^T)^{k'}(\xi+\alpha_{i'}))} = \delta_{kk'}\delta_{ii'} \tag{13}$$

where $k,k'=0,1$, $i,i'=0,\ldots,7$ and α_i are the coset representatives of $A^{-1}\mathbb{Z}^2/\mathbb{Z}^2$ (that is: $\alpha_0=(0,0),\alpha_1=(\frac{1}{4},0),\alpha_2=(\frac{1}{2},0),\alpha_3=(\frac{3}{4},0),\alpha_4=(0,\frac{1}{2}),\alpha_5=(\frac{1}{4},\frac{1}{2}),\alpha_6=(\frac{1}{2},\frac{1}{2}),\alpha_7=(\frac{3}{4},\frac{1}{2})$), and

$$\lim_{j\to\infty}\sum_{k\in\mathbb{Z}}|\hat{\phi}((B^T)^kA^{-j}\xi)|^2 = 1 \;\; a.e.\;\; \xi\in\widehat{\mathbb{R}}^2,$$

then $\psi^1,\ldots,\psi^L$ is a tight frame AB–multiwavelet.

This theorem generalizes a similar result in the classical MRA theory.[2] In particular, equation (13) is the analog of the Smith–Barnwell equation that describes a so-called *perfect reconstruction* condition in the theory of filter banks.

This approach also leads to the following recursive algorithm for the computation of the AB-wavelet coefficients $\langle f,\psi_{ijk}^\ell\rangle$, where $\psi_{ijk}^\ell = D_A^iD_B^jT_k\,\psi^\ell$, that generalizes the classical *cascade algorithm* for wavelets.

Suppose that $f\in V_1$, then

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2}\langle f, D_A^{-1}D_B^jT_k\phi\rangle\,D_A^{-1}D_B^j\,T_k\,\phi.$$

In addition, if we assume (12), where $\psi^0=\phi$ and m_0^ℓ and m_1^ℓ satisfy (13), then we have

$$\psi^\ell = \sum_{k\in\mathbb{Z}^2} h^\ell(k)D_A^{-1}T_k\,\phi + \sum_{k\in\mathbb{Z}^2} g^\ell(Bk)D_A^{-1}D_BT_k\,\phi,$$

where

$$m_0^\ell(\xi) = \frac{1}{2\sqrt{2}}\sum_{k\in\mathbb{Z}^2} h^\ell(k)e^{-2\pi ik\cdot\xi}, \quad m_1^\ell(\xi) = \frac{1}{2\sqrt{2}}\sum_{k\in\mathbb{Z}^2} g^\ell(Bk)e^{-2\pi ik\cdot\xi}.$$

Letting $d_j^\ell(k) = \langle f, D_B^j T_k\, \psi^\ell \rangle$ and $c_j(k) = \langle f, D_A^{-1} D_B^j T_k\, \phi \rangle$, for $\ell = 0, \ldots, L$, we have the **analysis equation**:

$$d_j^\ell(k) = \sum_{m\in\mathbb{Z}^2} h^\ell(m - Ak)\, c_{2j}(m) + \sum_{m\in\mathbb{Z}^2} g^\ell(m - Ak)\, c_{2j-1}(B^{-1}m). \tag{14}$$

A similar argument gives the corresponding **synthesis** or **reconstruction equations**:

$$c_{2j}(k) = \sum_{\ell=0}^{L} \sum_{m\in\mathbb{Z}^2} h^\ell(k - Am)\, d_j^\ell(m) \tag{15}$$

and

$$c_{2j-1}(k) = \sum_{\ell=0}^{L} \sum_{m\in\mathbb{Z}^2} g^\ell(k - B^{-1}Am)\, d_j^\ell(m). \tag{16}$$

4. CONCLUSION

We have presented a new class of multidimensional representations obtained from the action of translations, dilations, and shear transformations on a finite set of generators in $L^2(\mathbb{R}^2)$. These representations exhibit exactly those mathematical and geometrical properties, including multiscale, localization, anisotropy, directionality, recently advocated by many authors for the construction of efficient image representations. One advantage of this approach is that these systems can be constructed using a generalized multiresolution analysis and implemented efficiently using an appropriate version of the classical cascade algorithm.

We are currently investigating the regularity issues associated with these systems, and the connection of our approach with some recent results about directional filter banks, such as, in particular, the curvelets and the contourlets[7, 8, 17]

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