Pointwise and directional regularity of nonharmonic Fourier series

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Abstract

We investigate how the regularity of nonharmonic Fourier series is related to the spacing of their frequencies. This is obtained by using a transform which simultaneously captures the advantages of the Gabor and Wavelet transforms. Applications to the everywhere irregularity of solutions of some PDEs are given. We extend these results to the anisotropic setting in order to derive directional irregularity criteria.

Key words:
PACS:

This paper is dedicated to the memory of Jean Morlet, who made me vividly aware of the importance of cross-fertilizations between mathematics and signal processing.

Jean Morlet always advocated the importance of comparing the possibilities of two key tools in signal processing: The wavelet transform and the Gabor transform. One purpose of the present paper is to show on a explicit problem that these tools should not be opposed, but can be combined in order to yield optimal results in the study of the regularity of *nonharmonic Fourier series*.

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$; a function which can be written

$$f(x) = \sum_{n\in\mathbb{N}} a_n e^{i\lambda_n\cdot x}, \tag{1}$$

(where $(a_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers) is called a nonharmonic Fourier series; note that the word "nonharmonic" points to the fact that we do not make the assumption that the frequencies λ_n are a subset of a discrete subgroup of $\mathbb{R}^d$; see [18] for for a comprehensive study of the properties of nonharmonic Fourier series. Our purpose is to study the relationship between the regularity of such series and the spacing of their frequencies.

The first example of a Fourier series displaying large gaps between its frequencies was the function

$$\mathcal{R}(t)=\sum_{n=1}^{\infty}\frac{\sin(n^2t)}{n^2}, \tag{2}$$

which Riemann expected to be a possible example of a continuous nowhere differentiable function, see [10,14]. The relationship between the spacings of the frequencies of a Fourier series and its everywhere irregularity has been the subject of many investigations, starting with the famous example of the Weierstrass functions; in Section 1, we review these results and show how they can all be derived from a simple property of the *Gabor-Wavelet transform*: Theorem 7 yields general irregularity results for multidimensional nonharmonic Fourier series. In Section 2, we show that, surprisingly, this simple result is optimal only for Hölder exponents less than 1, and Theorem 11 shows how it can be improved for larger Hölder exponents. Extensions are worked out in Section 3, and applications to everywhere irregularity of solutions of PDEs are derived in Section 4. In Section 5, the notion of directional irregularity is discussed, and a new definition is proposed; the techniques of Section 1 are adapted in a non-isotropic framework, and we derive directional irregularity results for Fourier series whose frequencies have lacunarities in certain directions.

1 Pointwise T^p_u irregularity of lacunary Fourier series

The notion of pointwise regularity most commonly used in statements concerning lacunary Fourier series is the *pointwise Hölder regularity*. The natural setting for this notion is supplied by functions *with slow growth*, i.e. locally bounded functions f which satisfy

$$\exists C,A>0,\ \ \forall R,\qquad \sup_{B(0,R)}|f(x)|\le C(1+R)^A,$$

where $B(x,R)$ denote the open ball centered at x and of radius R.

Definition 1 *Let $f:\mathbb{R}^d\to\mathbb{C}$ be a function with slow growth and let $\alpha\ge 0$; $f\in C^\alpha(x_0)$ if there exist $R>0$, $C>0$, and a polynomial P of degree less than α such that, for R small enough,*

$$\textit{if } |x-x_0|\le R,\quad \textit{then } |f(x)-P(x-x_0)|\le C|x-x_0|^\alpha. \tag{3}$$

The Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$.

This notion is only pertinent when applied to locally bounded functions; indeed, (3) implies that f is bounded in a neighbourhood of x_0. We will derive pointwise irregularity results for solutions of PDEs where the natural function space setting is L^p or a Sobolev space which includes unbounded functions. In such cases, one has to use an extension of pointwise smoothness, which was introduced by Calderón and Zygmund in 1961, see [3]. Let $B(x_0, R)$ denote the open ball centered at x_0 and of radius R. We first define the global setting fitted to this notion.

Definition 2 *Let $f :\to \mathbb{C}$ be a function which locally belongs to $L^p(\mathbb{R}^d)$. The function f has slow growth in L^p if*

$$\exists A, C > 0, \;\; \forall R, \qquad \int_{B(0,R)} |f(x)|^p dx \leq C(1+R)^A. \tag{4}$$

Definition 3 *Let $p \in [1, +\infty)$ and $\alpha > -d/p$. Let f be a function with slow growth in L^p. We say that f belongs to $T^p_\alpha(x_0)$ if there exist $C > 0$ and a polynomial P of degree less than α such that, for r small enough,*

$$\left(\frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq C r^\alpha. \tag{5}$$

The p-exponent of f at x_0 is $h^p_f(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}$.

Note that $h^\infty_f(x_0) = h_f(x_0)$ and, if $p \geq q$, then $h^p_f(x_0) \leq h^q_f(x_0)$, see [3].

1.1 A pointwise irregularity criterium

We start by establishing a general pointwise irregularity criterium based on the *Gabor-wavelet transform* (referred to in the following as the GW transform). We will use the following notations: If $\lambda, \; x \in \mathbb{R}^d$, $\lambda \cdot x$ denotes the usual scalar product of λ and x; if Ω is a rotation in $\mathbb{R}^d$ (i.e. belongs to SO_d), then

$$\phi_\Omega(x) = \phi(\Omega(x)).$$

Definition 4 *Let $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function in the Schwartz class such that $\hat{\phi}(\xi)$ is supported in the unit ball centered at 0. The GW transform of a function or a tempered distribution f defined on $\mathbb{R}^d$ is defined by*

$$d(a, b, \lambda, \Omega) = \frac{1}{a^d} \int_{\mathbb{R}^d} f(x) e^{-i\lambda \cdot x} \phi_\Omega\left(\frac{x-b}{a}\right) dx. \tag{6}$$

The following pointwise irregularity criterium is a variant of the usual wavelet criterium, either for Hölder regularity, see [9], or for T^p_α regularity, see [11,12].

Proposition 5 *Let $p \in (1, +\infty]$ and $\alpha > -d/p$. Let f be a function with slow growth in L^p; if f belongs to $T^p_\alpha(x_0)$, then there exists $C > 0$ such that*

$$\forall a \in (0,1], \ \ \forall b : |x_0 - b| \leq 1, \ \ \forall \Omega \in SO_d, \ \ \forall \lambda \ : \ |\lambda| \geq 1/a,$$

$$|d(a,b,\lambda,\Omega)| \leq Ca^\alpha \left(1 + \frac{|x_0 - b|}{a}\right)^{\alpha + d/p}. \tag{7}$$

Remark: If we pick λ of the form $\lambda = u/a$, where u is a fixed vector satisfying $\| u \| \geq 1$, then (7) boils down to the usual the two-microlocal condition $f \in C^{\alpha, -\alpha - d/p}(x_0)$; indeed, $\psi(x) = e^{-iux}\phi(x)$ has vanishing moments of all orders so that restricting to constants λ of this form in the GW transform amounts to consider the usual continuous wavelet transform of f. This two-microlocal condition essentially means that a fractional primitive of f of order d/p belongs to $C^{\alpha + d/p}(x_0)$, see [9,13] for precise statements and results, and [11] for sharper results in the case of the wavelet transform.

Proof of Proposition 5: If $\omega_{a,\lambda,\Omega}(x) = a^{-d} e^{-i\lambda \cdot x}\phi_\Omega(x/a)$, then

$$\widehat{\omega_{a,\lambda,\Omega}}(\xi) = \hat{\phi}(a\, \Omega^*(\xi + \lambda)),$$

so that, as soon as $|\lambda| > 1/a$, then $\widehat{\omega_{a,\lambda,\Omega}}$ and all its derivatives vanishe at 0. It follows that, if P is the polynomial given by (5), then

$$d(a,b,\lambda,\Omega) = \frac{1}{a^d} \int_{\mathbb{R}^d} (f(x) - P(x - x_0)) e^{i\lambda \cdot x} \phi_\Omega \left(\frac{x-b}{a} \right) dx.$$

For $n \geq 0$, let $B_n = B(b, 2^n a)$, $\Delta_n = B_{n+1} - B_n$ and $\Delta_0 = B_0$. We split $d(a,b,\lambda,\Omega)$ as a sum of integrals I_n over Δ_n. Let q denote the conjugate exponent of p; by Hölder's inequality,

$$\forall n \geq 0, \qquad |I_n| \leq \frac{1}{a^d} \parallel f(x) - P(x - x_0) \parallel_{L^p(B_{n+1})} \left\| \phi_\Omega \left(\frac{x-b}{a} \right) \right\|_{L^q(\Delta_n)}.$$

Note that, because of the global slow growth assumption, the precise value of r in (5) is irrelevant, and we can pick $r = 2$.

Let N be such that

$$2^N a \leq 1 < 2^{N+1} a.$$

If $n \leq N$, then all integrals bear on domains included in $B(x_0, 2)$. In that case, since $B_{n+1} \subset B(x_0, |x_0 - b| + 2^{n+1}a)$, and since ϕ has fast decay, for all D large enough,

$$|I_n| \leq \frac{CC'(D)}{a^d}(|x_0 - b| + 2^{n+1}a)^{\alpha+d/p}a^{d/q}(2^{-Dn})^{1/q}.$$

If $n > N$, we use the slow growth assumption on f, which yields that

$$|I_n| \leq C(2^n a)^A a^{d/q}(2^{-Dn})^{1/q},$$

where A is given, but D can be picked aribitrarily large (because of the fast decay of ϕ).

Summing up on all values of n, we obtain that $|d(a, b, \lambda, \Omega)|$ is bounded by

$$Ca^{-d/p}\sum_{n=0}^{N}(|x_0 - b| + 2^{n+1}a)^{\alpha+d/p}2^{-Dnq} + \sum_{N+1}^{\infty} C(2^n a)^A a^{d/q}(2^{-Dn})^{1/q}$$

$$\leq C\left(|x_0 - b|^{\alpha+d/p}a^{-d/p} + a^{\alpha}\right).$$

Hence Proposition 5 holds.

1.2 A first application to nonharmonic Fourier series

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$. We will consider series of the form given by (1). Of course, we can (and will) assume that the λ_n are distinct; note that we do not assume the λ_n to be integers. One refers to such series as *nonharmonic Fourier series*, see [18]. In order to perform a GW analysis of f, we need this series to converge in the space of tempered distributions. A straightforward sufficient condition for convergence in $\mathcal{S}'$ of (1) is

$$\exists N \in \mathbb{N}: \qquad \sum_{n\in\mathbb{N}} \frac{|a_n|}{(1 + |\lambda_n|)^N} < \infty. \tag{8}$$

Definition 6 *Let (λ_n) be a sequence in $\mathbb{R}^d$. The gap sequence associated with (λ_n) is the sequence (θ_n) defined by*

$$\theta_n = \inf_{m\neq n} |\lambda_n - \lambda_m|$$

(θ_n is the distance between λ_n and its closest neighbour).

We say that the sequence (λ_n) *is separated if*

$$\inf_n \theta_n > 0.$$

Note that (8) is not a necessary condition for the convergence of (1) in $\mathcal{S}'$; however, if the sequence (λ_n) is a finite union of separated sequences (which will always be the case in the applications that we will consider), then one easily checks that (8) is indeed a necessary and sufficient condition of convergence in $\mathcal{S}'$. Therefore, we will always assume in the following that (8) holds. The following statement includes, and slightly improves the previous results concerning everywhere irregularity of (nonharmonic) Fourier series.

Theorem 7 *Let f be given by (1), where we assume that the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ satisfy (8). Let x_0 be a given point of $\mathbb{R}^d$, $p>1$, $\alpha > -d/p$ and assume that f has slow growth in L^p. If $f \in T^p_\alpha(x_0)$, then there exists C such that, for all n,*

$$\text{if } |\lambda_n| \geq \theta_n, \qquad \text{then} \qquad |a_n| \leq \frac{C}{(\theta_n)^\alpha}. \tag{9}$$

Thus, if

$$H = \sup\{\alpha : \ (9) \text{ holds}\},$$

then, for any $x_0 \in \mathbb{R}^d$, $h^p_f(x_0) \leq H$ (hence $h_f(x_0) \leq H$).

Note that previous results anticipating Theorem 7 only dealt with pointwise Hölder regularity. The following definition supplies a strong lacunarity condition, often considered in the past.

Definition 8 *Let $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^d$; it is a Hadamard sequence if it is separated and satisfies*

$$\exists C > 0 \quad \text{such that} \qquad \forall n, \quad \theta_n \geq C|\lambda_n|. \tag{10}$$

Many papers have dealt with irregularity properties of lacunary Fourier series. We only mention a few landmarks that anticipated Theorem 7.

- In 1962, G. Freud considered one-dimensional Hadamard sequences of integer in dimension 1, see [5]. He showed that, if $f \in L^1(T)$, then h_f is constant.
- In 1965, M. Izumi, S.-I. Izumi and J.-P. Kahane obtained Theorem 7 in the following case: $d=1$, $\lambda_n \in \mathbb{Z}$, f is continuous and $f \in C^\alpha(x_0)$, see [8].

- The first multidimensional result was obtained by J. Pesek in 1977, who considered periodic multiple Fourier series, i.e. when $\lambda_n \in \mathbb{Z}^d$, see [17] (and also [14] for related results). Assume that $f \in L^1(T^d)$, $f \in C^\alpha(x_0)$ and let $\omega \in (0,1)$; J. Pesek showed that, if

$$\forall n, \quad \theta_n \geq C|\lambda_n|^\omega, \tag{11}$$

  then $a_n = O\left(|\lambda_n|^{-\omega\alpha}\right)$; note that the assumptions of Theorem 7 actually do not require the lacunarity condition to hold uniformly for all the λ_n: in order to recover an irregularity result, one only needs the lacunarity condition to hold on a subsequence.
- In 2006, J. Dixmier, J.-P. Kahane and J.-L. Nicolas proved Theorem 7 when the λ_n form a Hadamard sequence, see [4].

Proof of Theorem 7: It is a direct consequence of Proposition 5: We estimate the GW transform of f at particular points, and for a function ϕ such that $\hat{\phi}(\xi)$ is radial, supported in the unit ball centered at 0, and such that $\hat{\phi}(0) = 1$. Taking $\Omega = Id$, we consider

$$D_m = d\left(\frac{1}{\theta_m}, x_0, \lambda_m, Id\right). \tag{12}$$

On one hand,

$$\begin{aligned} D_m &= (\theta_m)^d \int \left(\sum_n a_n e^{i(\lambda_n - \lambda_m)\cdot x} \phi(\theta_m(x - x_0))\right) dx \\ &= \sum_n a_n \hat{\phi}\left(\frac{\lambda_m - \lambda_n}{\theta_n}\right) e^{i(\lambda_n - \lambda_m)\cdot x_0}; \end{aligned} \tag{13}$$

since $\hat{\phi}$ vanishes outside of $B(0,1)$, the definition of θ_n implies that $\hat{\phi}\left((\lambda_m - \lambda_n)/\theta_n\right) = \delta_{n,m}$, so that $D_m = a_m$. On the other hand, if $f \in T^p_\alpha(x_0)$, then Proposition 5 implies that, for any m such that $|\lambda_m| \geq \theta_m$, $|D_m| \leq C\theta_m^{-\alpha}$; Theorem 7 follows.

2 The Optimality of Theorem 7

J. Pesek showed that, for a given sequence of frequencies $\lambda_n \in \mathbb{Z}^d$, and for given $\alpha \in (0,1]$, and $\omega \in (0,1]$, if (11) does not hold, then there exists a sequence (a_n) such that $a_n \neq \mathcal{O}\left(|\lambda_n|^{-\omega\alpha}\right)$, though $f \in L^1(T^d)$ and $f \in C^\alpha(x_0)$. We now extend Pesek's optimality theorem in the following direction: the condition $\lambda_n \in \mathbb{Z}^d$ is not required (which means that its proof does not require the use of explicit trigonometric polynomials).

Proposition 9 *Let λ_n be a sequence taking values in $\mathbb{R}^d$. Let $\alpha \in (0,1)$ and $\omega \in (0,1]$. Assume that there exists a subsequence $\lambda_{n(m)}$ such that (11) does not hold; then there exists a sequence (a_n) such that $a_n \neq O\left(|\lambda_n|^{-\omega\alpha}\right)$, but $f \in L^1_{loc}$ and $f \in C^\alpha(x_0)$.*

Proof of Proposition 9: It is clearly sufficient to construct a one-dimensional example, and one can restrict to the case $x_0 = 0$. After perhaps extracting a subsequence, we can assume that λ_n has a subsequence composed of two sets of frequencies μ_n and ν_n which both increase at least exponentially, and satisfy

$$|\mu_n - \nu_n| \leq \varepsilon_n |\mu_n|^\omega$$

where (ε_n) tends to 0. Let γ_n be a sequence satisfying

- γ_n is decreasing and tends to 0
- $\gamma_n \geq \varepsilon_n$
- $\gamma_{n+1}/\gamma_n \to 1$.

Let

$$f(x) = \sum_n \frac{1}{\gamma_n^\alpha |\mu_n|^{\omega\alpha}} \left(e^{i\mu_n x} - e^{i\nu_n x}\right). \tag{14}$$

Note that $a_n \neq \mathcal{O}\left(|\lambda_n|^{-\omega\alpha}\right)$. Proposition 9 will be proved if we can show that $f \in C^\alpha(x_0)$.

The assumptions on γ_n and the exponential growth of μ_n imply that the sequence $1/(\gamma_n|\mu_n|^\omega)$ is decreasing for n large enough and tends to 0. Let N be such that

$$\frac{1}{\gamma_{N+1}|\mu_{N+1}|^\omega} < |x| \leq \frac{1}{\gamma_N|\mu_N|^\omega}.$$

Then

$$|f(x)| \leq \sum_{n=1}^{N} \frac{1}{\gamma_n^\alpha|\mu_n|^{\omega\alpha}} |\mu_n - \nu_n||x| \; + \; 2 \sum_{n=N+1}^{\infty} \frac{1}{\gamma_n^\alpha|\mu_n|^{\omega\alpha}}$$

$$\leq \sum_{n=1}^{N} \gamma_n^{1-\alpha}|\mu_n|^{\omega(1-\alpha)}|x| \; + \; 2 \sum_{n=N+1}^{\infty} \frac{1}{\gamma_n^\alpha|\mu_n|^{\omega\alpha}}.$$

Once again, the assumptions on γ_n and the exponential growth of μ_n imply that the sequence $\gamma_n^{1-\alpha}|\mu_n|^{\omega(1-\alpha)}$ (respectively $\gamma_n^{-\alpha}|\mu_n|^{-\omega\alpha}$) increases (respectively decreases) geometrically; therefore

$$|f(x)| \leq C\gamma_N^{1-\alpha}|\mu_N|^{\omega(1-\alpha)}|x| + C\frac{1}{\gamma_N^{\alpha}|\mu_N|^{\omega\alpha}},$$

which, using the definition of N is bounded by $C|x|^\alpha$.

J. Pesek expected his optimality result to be true without the limitation $\alpha < 1$ (and he actually claimed that, in dimension 1, his proof extends to larger values of α). Surprisingly, we will now show that, despite this natural expectation, Theorem 7 is not optimal when α is larger than 1.

Definition 10 *Let (λ_n) be a sequence in $\mathbb{R}^d$, with gap sequence (θ_n). The second order gap sequence $(\omega_n)_{n\in\mathbb{N}}$ associated with (λ_n) is the distance between λ_n and its second closest neighbour; it can be formally defined as follows: Let*

$$A_n = \{m : |\lambda_n - \lambda_m| = \theta_n\}.$$

If A_n has only one element $l(n)$, then $\omega_n = \inf_{m\notin\{n,l(n)\}} |\lambda_n - \lambda_m|$; else $\omega_n = \theta_n$.

Note that $\theta_n \leq \omega_n$. The following result improves theorem 7 when $\theta_n = o(\omega_n)$ and $\alpha > 1$; therefore it shows that, in such cases, Theorem 7 is not optimal.

Theorem 11 *Let f be given by (1), where we assume that the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ satisfy (8). Let x_0 be a given point of $\mathbb{R}^d$, $p > 1$, and assume that f belongs to L^p in a neighbourhood of x_0. Let $\alpha \geq 1$; if $f \in T^p_\alpha(x_0)$, then there exists C such that, for all n,*

$$\text{if } |\lambda_n| \geq \omega_n, \qquad \text{then} \qquad |a_n| \leq \frac{C}{(\omega_n)^{\alpha-1}\theta_n}. \tag{15}$$

Proof of Theorem 11: First, note that, if θ_n is not a $o(\omega_n)$, then Theorem 11 boils down to Theorem 7; so, we can assume that $\theta_n = o(\omega_n)$.

We apply again Proposition 5, but instead of the previous function ϕ, we use ϕ_1 defined by $\hat{\phi}_1(\xi) = \hat{\phi}(\xi)\xi_1$, where ξ_1 denotes the first component of ξ and $\hat{\phi}(\xi)$ is radial, supported in the unit ball centered at 0, and such that $\hat{\phi}(\xi) = 1$ in a neighbourhood of 0. Denote by Ω_m a rotation which maps the first canonical vector e_1 to the unit vector in the direction of $\lambda_m - \lambda_{l(m)}$, and let

$$E_m = d\left(\frac{1}{\omega_m}, x_0, \lambda_m, \Omega_m\right).$$

A similar computation to the one which led to (13) now yields $D_m = a_m|\lambda_{l(m)} - \lambda_m|/\omega_m$; Proposition 5 implies that $|D_m| \leq C\omega_m^{-\alpha}$; since $|\lambda_{l(m)} - \lambda_m| = \theta_m$, Theorem 7 follows.

One can easily prove the optimality of Theorem 11 when $1 \leq \alpha \leq 2$, using a function f similar to (14), but which involves differences of order two of complex exponentials. Note also that one could push the argument further and obtain more complicated criteria that would be optimal for larger and larger values of α.

An unexplored open problem is to understand the optimality of these criteria when both the sequence λ_n and the order of magnitude of the a_n are given. Let us give a simple example: Let (a_n) be a sequence of coefficients satisfying

$$\exists C, C' > 0 \quad \text{such that} \qquad \forall n \geq 1, \quad \frac{C}{n^2} \leq |a_n| \leq \frac{C'}{n^2}, \tag{16}$$

and let

$$f(t) = \sum_{n=1}^{\infty} a_n \sin(n^2 t). \tag{17}$$

Proposition 9 or Theorem 11 yield that the Hölder exponent of f is everywhere smaller than 2. In the case where $a_n = 1/n^2$, the function f considered is (2), and its largest Hölder exponent is 3/2, see [10]; however, it is not known if this is best possible; i.e. does there exist a sequence (a_n) satisfying (16) and such that the Hölder exponent of f at some points is larger than 3/2? Can it be as large as 2? (This is expected if Theorem 11 is optimal in that case.)

3 Applications and extensions of Theorems 7 and 11

3.1 Hölder range of (q, δ) lacunary nonharmonic Fourier series

We first apply Theorems 7 and 11 to a class of lacunary nonharmonic Fourier series whose frequencies grow polynomially, as in (2).

Definition 12 *Let f be given by (1), and let δ be defined by*

$$\delta = \liminf \left(\frac{-\log |a_n|}{\log |\lambda_n|} \right).$$

Assume that δ is finite and that, for some $q > 1$,

$$\forall n, \qquad \theta_n \geq C|\lambda_n|^{(q-1)/q}, \tag{18}$$

then f is called a (q, δ)-series.

The series (2) is an example of $(2, 1)$-series. Typical examples of (q, δ)-series are supplied by

$$\sum R(n) e^{i(Q(n)x)},$$

where R and Q are rational fractions, and $q = deg(Q)$, $\delta = -deg(R)/deg(Q)$ (R and Q can also be rational functions of n, $\log n$, $\log\log n, \ldots$ in which case, only the degrees in the variable n play a role).

Definition 13 *Let $f \in L^\infty_{loc}(\mathbb{R}^d)$. The Hölder range of f is*

$$\mathcal{H}_R(f) = \{H : \ \exists x_0 \in \mathbb{R}^d : h_f(x_0) = H\}.$$

A function f is a monohölder function if its Hölder range is reduced to one point, i.e. if its Hölder exponent is constant.

Results on the Hölder range will make use of the following uniform regularity criteria.

Lemma 14 *Let f be an L^1_{loc} function given by (1) with $\lambda_n \in \mathbb{R}^d$.*

Let

$$\theta = \sup\{\omega : a_n |\lambda_n|^\omega \in l^1\}.$$

Then, for all $\omega < \theta$, the function f belongs to $C^\omega(\mathbb{R}^d)$.

If there exist $a, b > 0$ such that

$$\sum_{n=1}^{N} |a_n \lambda_n| = O(N^a) \ \text{ and } \sum_{n=N}^{\infty} |a_n| = O\left(\frac{1}{N^b}\right), \tag{19}$$

then f belongs to $C^{b/(a+b)}(\mathbb{R}^d)$.

Proof of Lemma 14: Assume that $\theta \leq 1$. Let $\omega < \theta$; then $|e^{i\lambda_n . x} - e^{i\lambda_n . y}|$ is obviously bounded by 2, and (using the mean value theorem) by $|\lambda_n||x-y|$, hence by $2|x-y|^\omega |\lambda_n|^\omega$; therefore

$$|f(x) - f(y)| \leq 2|x-y|^\omega \sum |a_n||\lambda_n|^\omega \leq C|x-y|^\omega.$$

The case $\theta > 1$ follows by taking partial derivatives of the series (1) up to the order $[\theta]$. The proof of the second part is similar: Let $x, y \in \mathbb{R}^d$ and N be defined by

$$\frac{1}{N^{a+b}} \leq |x-y| < \frac{1}{(N-1)^{a+b}}.$$

In the estimate of $|f(x)-f(y)|$, one bounds the increment $|e^{i\lambda_n.x} - e^{i\lambda_n.y}|$ either by $|\lambda_n||x-y|$ if $n \leq N$ or by 2 else.

Note that the second part of Lemma 14 also extends to regularity exponents larger than 1 by derivating the series.

The following corollary gives a quantitative content to the intuitive rule that the more lacunary the Fourier series, the smaller its Hölder range.

Corollary 15 *Let f be given by (1).*

- *If there exists a subsequence n_k such that $|a_{n_k}| \geq C|\lambda_{n_k}|^{-\delta}$ and $\theta_{n_k} \geq C|\lambda_{n_k}|^{(q-1)/q}$, then*

$$\forall x \in \mathbb{R}^d, \qquad h_f(x) \leq \delta q/(q-1).$$

- *If f is a (q,δ)-series with $\delta > d/q$, its Hölder range is contained in the interval $\left[\delta - \frac{d}{q}, \frac{\delta q}{q-1}\right]$.*

Remark: When $q = +\infty$ (i.e. if the sequence (λ_n) has fast growth), this result boils down to a classical monohölder property of such Fourier series (hence also valid for Hadamard series):

$$\text{Let} \qquad \delta = \liminf \left(\frac{-\log(a_n)}{\log(\lambda_n)} \right); \qquad \text{if } \delta > 0, \text{ then} \qquad \forall x \;\; h_f(x) = \delta.$$

Proof of Corollary 15: The first statement follows from Theorem 7. Denote by Λ_j the shell

$$\Lambda_j = \{\xi : 2^j \leq |\xi| < 2.2^j\}.$$

If (λ_n) satisfies (18), then the number of frequencies λ_n in Λ_j is bounded by

$$N_j = C.\frac{2^{dj}}{(2^j)^{d(q-1)/q}} = C.2^{dj/q}.$$

In order to use Lemma 14, we split the series $\sum |a_n|||\lambda_n|^\omega$ into sums over each shell. The sum for $\lambda_n \in \Lambda_j$ is bounded by $2^{dj/q}2^{-\delta j}2^{-\omega j}$, and therefore, the whole series converges if $\omega < \delta - d/q$. Hence the lower bound for the Hölder exponent holds.

A last implication concerns *chirp exponents* of lacunary Fourier series. We start by recalling the definition of a chirp, see [13]. Let f be a tempered distribution; fractional primitives of f can be defined as follows: Let $\mathcal{F}$ denote the Fourier transform; I^s is the operator satisfying

$$\mathcal{F}(I^s(f))(\xi) = (1+|\xi|^2)^{-s/2}\mathcal{F}(f)(\xi).$$

Definition 16 *Let f be a locally bounded function with slow growth; f is a chirp of type (h,γ) at x_0 if*

$$\forall s \geq 0, \qquad I^s(f) \in C^{h+s(\gamma+1)}(x_0). \tag{20}$$

The chirp exponent of f at x_0 is

$$\beta_f(x_0) = \sup\{\gamma : \ (20) \quad \textit{holds for} \quad h=0\}.$$

Remark: It is sufficient to check that (20) holds for $s=0$ and for a sequence s_n which tends to $+\infty$, see [13].

Typical examples of chirps (in dimension $d=1$) are supplied by the functions

$$f_{\alpha,\beta}(x) = |x|^\alpha \sin\left(\frac{1}{|x|^\beta}\right),$$

whose Hölder exponent at 0 is α and chirp exponent is β, see [13].

Note that $(Id - \Delta^2)e^{i\lambda\cdot x} = (1+|\lambda|^2)e^{i\lambda\cdot x}$. It follows that, if f is a (q,δ) lacunary Fourier series, then, it can be written as a fractional derivative of order 2 of a $(q,\delta+2)$ lacunary Fourier series, and therefore (using the remark which follows Definition 16) the Hölder exponent of its fractional primitive of order 2 will be at most $(\delta+1)q/(q-1)$. Using this argument for primitives of arbitary order, one obtains the following conclusion.

Corollary 17 *The chirp exponent of a (q,δ) lacunary Fourier series is at most $1/(q-1)$.*

Two historical examples show that this result is sharp: (2) has a chirp of exponent 1, see [13], and D. Boichu proved that $\sum n^{-3}\sin(n^3 t)$ has chirps of exponent $1/2$, see [2].

3.2 Additional irregularity criteria

We will now show the flexibility of the method used in the proof of Theorem 7 by obtaining additional irregularity criteria which depend either on non-cancellation properties of the coefficients a_n or on particular spacings of the frequencies. We start by defining the *gap sequences of arbitrary order* associated with a sequence λ_n.

Definition 18 *Let $\lambda_n \in \mathbb{R}^d$, and let k be a positive integer. The gap sequence of order k asociated with λ_n is the sequence θ_n^k such that θ_n^k is the distance between λ_n and its k-th closest neighbour (counted with multiplicity).*

The set A_n^k is composed of the k-closest neighbours of λ_n (including λ_n), and $B_n^k = A_n^k - \{\lambda_n\}$.

Remarks: The sets A_n^k and B_n^k are not well defined if several frequencies are at the same distance with λ_n; however, in that case, we can make an arbitary choice among these frequencies; it will not have any incidence on the following results. Note that $\theta_n^1 = \theta_n$ and $\theta_n^2 = \omega_n$.

Proposition 19 *Let f be given by (1), where the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ satisfy (8). Let $\omega < 1$ be given, and assume that n and k are such that $\theta_n^{k-1} \le \omega\theta_n^k$. Let x_0 be a given point of $\mathbb{R}^d$, $p > 1$ and $R > 0$; if $f \in T^p_\alpha(x_0)$, then*

$$\exists C > 0,\ \forall b \in B\left(x_0, \frac{R}{\theta_n^k}\right), \qquad \left|\sum_{l\in A_n^k} a_l e^{i\lambda_l\cdot b}\right| \le \frac{C(1+R)^{\alpha+d/p}}{(\theta_n^k)^\alpha}. \tag{21}$$

If furthermore there exists $A > 0$ such that

$$|a_n| \ge (1+A)\left|\sum_{l\in B_n^k} a_l e^{i\lambda_l\cdot b}\right|, \qquad \textit{then} \qquad |a_n| \le \frac{C}{(\theta_n^k)^\alpha}.$$

Proof of Proposition 19: The proof is similar to the proof of Theorem 7, but requires a function ϕ which satisfies the stronger requirement

$$supp(\hat{\phi}) \subset B(0,1) \ \text{ and } \qquad \forall \xi \ \text{ such that } |\xi| \le \omega, \quad |\hat{\phi}(\xi)| = 1. \tag{22}$$

Let $D_n = d\left(\frac{1}{\theta_n^k}, b, \lambda_n, Id\right)$. On one hand, since $\hat{\phi}$ satisfies (22), the same argument that led to (13) now yields

$$D_n = \sum_l a_l \hat{\phi}\left(\frac{\lambda_l - \lambda_n}{\theta_n^k}\right) e^{i(\lambda_l-\lambda_n)\cdot b} = \sum_{l \in A_{n_k}} a_l e^{i(\lambda_l-\lambda_n)b}. \tag{23}$$

On the other hand, if $f \in T^p_\alpha(x_0)$, then Proposition 5 implies that $|D_n| \leq C(1+R)^{\alpha+d/p}(\theta_n^k)^{-\alpha}$; the second part of Proposition 19 follows.

The third part follows immediately, since

$$\left|\sum_{l\in A_n^k} a_l e^{i\lambda_l \cdot b}\right| \geq |a_n| - \left|\sum_{l \in B_n^k} a_l e^{i\lambda_l\cdot b}\right| \geq \frac{A}{A+1}|a_n|.$$

The second part of Proposition 19 yields the same conclusion as Theorem 7 in settings where no separation condition between the frequencies holds. Unlike Theorems 7 and 11, the first part can give a different upper bound of the Hölder exponent at different points.

In order to illustrate the different ranges of applications of Theorem 7, Theorem 11 and Proposition 19 we compare them on the toy-example supplied by the series

$$f_\alpha(x) = \frac{1}{2}\sum_{j=0}^\infty 2^{-\alpha j}\left[\cos((2^j-1)x) - \cos((2^j+1)x)\right] = \sin x \sum_{j=0}^\infty 2^{-\alpha j}\sin(2^j x),$$

where $\alpha \in (0,1)$. Indeed, in that case, the local exponents can be computed by hand and one immediately checks that $h_{f_\alpha}(k\pi) = \alpha+1$, and everywhere else: $h_{f_\alpha}(x) = \alpha$. As a test of the respective efficiencies of the previous results, let us check what they yield in this situation.

The frequencies of this series are a union of two Hadamard series. Since forall n, one has $\theta_n = 2$, Theorem 7 does not yield any irregularity result. Since $\omega_n = 2^{j-1}-2$, Theorem 11 allows to capture the largest regularity index: It states that f_α is nowhere $C^{\alpha+1}$. Finally, Proposition 19 allows to discriminate between behaviors at different points: Taking $R = 0$ in (21) yields that, if $f_\alpha \in C^\beta(x_0)$, then $2^{-\alpha j}|e^{i(2^j-1)x_0} - e^{i(2^j+1)x_0}| \leq C2^{-\beta j}$; which implies the sharp upper bound for the Hölder exponent at every point.

4 Everywhere irregularity of solutions of PDEs

As a consequence of the previous results, we will show that the solutions of some PDEs display a remarkable property of everywhere irregularity if the initial condition is not smooth. Our purpose is not to obtain results in their most general form but, through some toy-examples, to point out unexpected applications of Theorem 7 in the field of PDEs.

4.1 Schrödinger's equation in one dimension

Consider the one-dimensional Schrödinger equation without potential

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2}, \qquad \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R} \tag{24}$$

$$\text{with initial condition:} \quad \psi(x,0) = \psi_0(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}. \tag{25}$$

We first assume that ψ_0 is a periodic distribution, i.e. that the sequence a_n increases at most polynomially. Let us make precise the way by which its initial value can be defined. If $u(x)$ and $\phi(t)$ are two functions, $(u \otimes \phi)(x,t)$ denotes the function $u(x)\phi(t)$.

Definition 20 *Let $t_0 \in \mathbb{R}$; let $\phi \in \mathcal{D}(\mathbb{R})$ be such that $\int \phi(x)dx = 1$. Let $\phi_\lambda(t) = \frac{1}{\lambda}\phi\left(\frac{t-t_0}{\lambda}\right)$; the trace of a two-variables distribution ω at time t_0 (if it exists) is the distribution ω_{t_0} such that*

$$\forall u \in \mathcal{D}(\mathbb{R}) \qquad \langle \omega_{t_0} | u \rangle = \lim_{\lambda \to 0} \langle \omega | u \otimes \phi_\lambda \rangle.$$

The general solution of (24) can be written

$$\psi(x,t) = \sum_{n \in \mathbb{Z}} a_n e^{inx} e^{-in^2 t}, \tag{26}$$

which means that (26) has the following properties:

- It is indeed a solution of (24) in the sense of distributions.
- It has a trace ψ_{t_0} at any time $t_0 \in \mathbb{R}$, in the sense of Definition 20, and at time $t_0 = 0$, its value is given by (25).
- The trace ψ_{t_0} is a continuous function of t_0, i.e. $t_0 \to \psi_{t_0}$ belongs to $\mathcal{C}(\mathbb{R}, \mathcal{D}')$; furthermore, if $\psi_0 \in H^s$, for an $s \in \mathbb{R}$, then $t_0 \to \psi_{t_0}$ belongs to $\mathcal{C}(\mathbb{R}, H^s)$.

Note that (26) is of the form $\sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n\cdot X}$ where $\lambda_n = (n, -n^2)$, and $X = (x,t)$, so that the gap sequence θ_n associated with λ_n satisfies

$$\forall n \in \mathbb{Z}, \quad \theta_n \geq |n| + 1. \tag{27}$$

Let $p > 1$ and $\alpha \geq -2/p$. It follows from Theorem 7 that

$$\text{if} \quad \psi \in T^p_\alpha(x_0, t_0), \quad \text{then} \qquad \forall n, \qquad |a_n| \leq \frac{C}{(|n|+1)^\alpha}. \tag{28}$$

But, if $|a_n| \leq C(|n|+1)^{-\alpha}$, then $\psi_0(x)$ belongs to the periodic Sobolev space H^s, as soon as $s < \alpha - 1/2$. (Note that Theorem 11 does not yield any improvement in this case.)

One can also consider the trace $\tilde{\psi}_{x_0}$ of ψ at a given point x_0, as a function of t, i.e. formally, $\tilde{\psi}_{x_0}(t) = \psi(x_0, t)$. It is again well defined in the sense of Definition 20, and the solution is still a one-dimensional lacunary Fourier series with $\lambda_n = n^2$. we obtain that, if $\tilde{\psi}_{x_0}(t) \in T^1_\alpha(t_0)$, then $|a_n| \leq C/n^\alpha$. Hence the following corollary holds.

Corollary 21 *Let $s > -5/2$, let $\psi(x,t)$ be a solution of (24), and assume that $\psi_0 \notin H^s$. Then*

$$\forall \alpha > s + 1/2, \ \forall (x_0, t_0), \ \forall p > 1, \qquad \psi \notin T^p_\alpha(x_0, t_0).$$

Furthermore, as regards irregularity in the time direction,

$$\forall \alpha > s + 1/2, \ \forall (x_0, t_0), \ \forall p > 1, \qquad \psi_{x_0} \notin T^p_\alpha(t_0).$$

In particular, when (24) has a bounded solution, then $\forall \alpha > s + 1/2, \forall (x_0, t_0)$, $\psi_{x_0} \notin C^\alpha(t_0)$.

Remarks: This is an irregularity result either in all directions or in the time direction. The solutions can of course be smooth for a given t_0 in the x direction. For example, assume that the initial condition is piecewise smooth; the solution at time $t = 2\pi$ reproduces the initial condition, and is therefore piecewise smooth in the x direction.

The everywhere irregularity of solutions of the Schrödinger equation somehow means that their graph is a "fractal"; such properties for the graph of the fundamental solution of (24) have been investigated by K. Oskolkov, see [16] and references therein.

4.2 Simply supported beams

A closely related equation is supplied by the vibrations of simply supported beams:

$$\text{if} \quad (x,t) \in [0,\pi] \times \mathbb{R}^+, \quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0, \tag{29}$$

$$\text{if} \quad x \in [0,\pi], \quad u(x,0) = u_0(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \tag{30}$$

$$\text{if} \quad x = 0, \quad \text{and} \quad x = \pi, \quad u(0,t) = u(\pi,t) = 0. \tag{31}$$

The general solution can formally be written

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n^2 t) + b_n \sin(n^2 t)) \sin(nx),$$

where the a_n and b_n are defined by: $u_0(x) = \sum a_n \sin(nx)$, and $u_1(x) = \sum n^2 b_n \sin(nx)$. The solution is of the same type as (26), except that we have to add the frequencies $\mu_n = (n, n^2)$; however, the associated gap sequence θ_n is the same. Denote by u_{x_0} the solution at time t_0; since the beam equation makes sense only if the solution is continuous, we assume that $u_0 \in H^{1/2}$ and $u_1 \in H^{-3/2}$. The same arguments developped for the Schrödinger equation yield the following result.

Corollary 22 *Let $s > 1/2$, let $u(x,t)$ be a solution of (29), and assume that either $u_0 \notin H^s$ or $u_1 \notin H^{s-2}$. Then*

$$\forall \alpha > s + 1/2, \ \forall (x_0,t_0), \ \ \forall p > 1, \quad u \notin T^p_\alpha(x_0,t_0), \quad \textit{and} \quad u_{x_0} \notin T^p_\alpha(t_0).$$

4.3 Schrödinger's equation in two dimensions and vibrating plates

In the previous cases, all frequencies satisfy the uniform separation condition (27); therefore, if the solution is irregular at one point, then all coefficients a_n have to be large. We will now consider cases where the separation condition only holds for *subsequences* of frequencies. In that case, smoothness at one point only implies that the the corresponding subsequence of the a_n is small, and one cannot infer everywhere irregularity results for *all* possible irregular initial condition, as in Corollaries 21 and 22. However, some initial conditions must be smooth, and therefore, by contraposition *some* initial conditions generate everywhere irregular solutions. Though we won't investigate this topic which leads to intricate problems related with number theory, we just work out two simple two-dimensional examples.

Consider the two-dimensional periodic Schrödinger equation

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} = -\Delta\psi, \\ \\ \psi(x,y,0) = \psi_0(x,y) = \displaystyle\sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} e^{i(\frac{nx}{A}+\frac{my}{B})}. \end{cases} \tag{32}$$

We consider distribution solutions in the same sense as in Section 4.1. The general solution is

$$\psi(x,y,t) = \sum a_{n,m} e^{i(\frac{nx}{A}+\frac{my}{B})} e^{-i(\frac{n^2}{A^2}+\frac{m^2}{B^2})t} = \sum a_{n,m} e^{i\lambda_{n,m}\cdot X},$$

where $\lambda_{n,m} = (\frac{n}{A}, \frac{m}{B}, -\frac{n^2}{A^2} - \frac{m^2}{B^2})$, and $X = (x,y,t)$. After an affine transformation, this sequence of frequencies is of the form $(n, m, an^2 + bm^2)$, with $a, b > 0$. The following lemma gives a lower bound for the corresponding gap sequence.

Lemma 23 *Let $\lambda_{n,m} = (n, m, an^2 + bm^2)$, with $a, b > 0$, and $(n,m) \in \mathbb{Z}^2$. For $m = n^2$,*

$$\exists C, \quad \forall n, \qquad \theta_{n,n^2} \geq C \parallel \lambda_{n,n^2} \parallel^{1/4} \tag{33}$$

Proof of Lemma 23: For couples (n,m) satisfying $m = n^2$, let us estimate

$$\Delta = (an^2 + bm^2) - a(n+p)^2 - b(m+q)^2, \qquad \text{where } (p,q) \neq (0,0).$$

In order to prove (33), we will check that there exists $C > 0$ such that either $|p|$, $|q|$ or $|\Delta|$ is larger than $C|n|$; indeed, this will imply that

$$\text{if } (p,q) \neq (0,0), \text{ then } \qquad \parallel \lambda_{n,n^2} - \lambda_{p,q} \parallel \geq C|n|;$$

since $\parallel \lambda_{n,n^2} \parallel$ is of the order of magnitude of n^4, (33) will follow.

Note that

$$\Delta = -2anp - 2bn^2q - ap^2 - bq^2,$$

where p and q are integers and do not vanish simultaneously.

First assume that $q = 0$. Then, since $p \neq 0$,

$$|\Delta| = |ap(2n+p)| \geq |a(2n+p)|$$

so that, either $|p| \geq |n|$ or $|\Delta| \geq |an|$, and (33) holds since $\| \lambda_{n,n^4} \| \sim n^4$.

Now, assume that $q \neq 0$. It is sufficient to prove that

$$|\Delta| + ap^2 + bq^2 \geq cn^2. \tag{34}$$

If $|p| \geq |\frac{bn}{2a}|$, the result holds. Else, the left hand side of (34) is larger than $|b|n^2/2$.

It follows from Theorem 7 that, if $f \in T^p_\alpha(x_0, t_0)$, then $a_{n,n^2} = \mathcal{O}(n|^{-\alpha})$. The following corollary thus follows.

Corollary 24 *let ψ be a solution of (32), where the sequence a_{n,n^2} is not a $\mathcal{O}(n|^{-\alpha})$. Then*

$$\forall (x_0, t_0), \ \forall p > 1, \ \forall \alpha > -d/p, \qquad \psi \notin T^p_\alpha(x_0, t_0).$$

Now, we consider the two-dimensional supported plate equation, in a rectangular domain $\Omega = [0, A] \times [0, B]$.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \Delta^2 u = 0, \\ u = \Delta u = 0 \ \text{ on } \ \partial\Omega, \\ u(x, y, 0) = u_0(x, y) \quad \text{and} \quad \dfrac{\partial u}{\partial t}(x, y, 0) = u_1(x, y). \end{cases} \tag{35}$$

The general solution can formally be written

$$u(x, y, t) =$$

$$\sum \sin\left(\frac{nx}{A}\right) \sin\left(\frac{my}{B}\right) \left[a_{n,m} \cos\left(\left(\frac{n^2}{A^2} + \frac{m^2}{B^2}\right) t\right) + b_{n,m} \sin\left(\left(\frac{n^2}{A^2} + \frac{m^2}{B^2}\right) t\right)\right]$$

$$= \sum c_l e^{i\lambda_l \cdot X},$$

where the λ_l take all values of the form $(\frac{n}{A}, \frac{m}{B}, \pm(\frac{n^2}{A^2} + \frac{m^2}{B^2}))$, and $X = (x, y, t)$. This is almost the same set of frequencies as in the Schrödinger case, and one immediately checks that it satisfies the same gap condition. The following result follows.

Corollary 25 *Let $u(x, y, t)$ be a solution of (35) with initial conditions*

$$u(x, y, 0) = \sum a_{n,m} \sin\left(\frac{nx}{A}\right) \sin\left(\frac{my}{B}\right),$$

$$\frac{\partial u}{\partial t}(x,y,0) = \sum b_{n,m}\left(\frac{n^2}{A^2}+\frac{m^2}{B^2}\right)\sin\left(\frac{nx}{A}\right)\sin\left(\frac{my}{B}\right).$$

If the sequence a_{n,n^2} is not a $o(|n|^{-\alpha})$ or if the sequence b_{n,n^2} is not a $o(|n|^{-\alpha})$, then

$$\forall (x_0,t_0), \quad \forall p>1, \quad \forall \alpha > -\frac{d}{p}, \qquad \psi \notin T^p_\alpha(x_0,t_0).$$

Remark: Clearly, many other subsequences than (n,n^2) would lead to the same conclusion. We picked it as an illustration, but it may be changed depending on the particular initial conditions considered.

5 Directional irregularity of lacunary Fourier series

We will now extend Theorem 7 in a non-isotropic setting in order to obtain directional irregularity results for multidimensional lacunary Fourier series. We start by defining and motivating the notion of directional regularity that will be used.

5.1 Directional regularity

Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a locally bounded function. The definition of pointwise regularity supplied by Definition 1 is uniform in all directions; in order to take into account directional behaviors, it is natural to define the Hölder regularity at x_0 in a direction $u \in \mathbb{R}^d - \{0\}$ as the Hölder regularity at 0 of the one variable function $t \to f(x_0+tu)$. This definition has several drawbacks which are the consequence of the fact that, actually, it is defined as the trace of f on a line, which is a set of vanishing measure. Indeed, one ultimately wants to deduce this directional smoothness from the coefficients of f on some particular sets of functions (ridgelets or curvelets for instance). These functions will have a support of non-empty interior, and therefore will take into account the values of f *around* the line considered. Therefore the definition of directional smoothness should include such information. However, in the asymptotic of small scales, the values taken into account should be localized more and more sharply around this line. These considerations motivate Definition 27 below. We start by extending the definition of the degree of a polynomial in an anisotropic setting.

Definition 26 *Let $\alpha = (\alpha_1, \cdots \alpha_d)$ be d-uple of positive real numbers. A polynomial P has degree less than α if each monomial component of P of the form*

$C_\alpha \cdot t_1^{a_1} \cdots t_d^{a_d}$ *satisfies*

$$\sum \frac{a_i}{\alpha_i} < 1.$$

In the isotropic case where all the α_i are equal, this definition boils down to the usual definition of the degree. Let $\mathcal{U} = (u_1, \cdots, u_d)$ be an orthonormal basis of $\mathbb{R}^d$; we denote by $(t_1, \cdots, t_d)$ the coordinates of $x - x_0$ on the basis $\mathcal{U}$, i.e.

$$x - x_0 = \sum_{i=1}^{d} t_i u_i. \tag{36}$$

We now give the definition of anisotropic smoothness, which was already introduced by M. Ben Slimane in [1] in the case where $\mathcal{U}$ is the canonical basis of $\mathbb{R}^d$.

Definition 27 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally bounded function with slow growth. Let $\alpha = (\alpha_1, \cdots \alpha_d)$ be a d-uple of nonnegative real numbers satisfying $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$. We say that the function f belongs to $C^\alpha(x_0,\mathcal{U})$ if there exists $C > 0$ and a polynomial P of degree lesss than α such that the coordinates $t = (t_1, \cdots, t_d) \in \mathbb{R}^d$ of $x - x_0$ on $\mathcal{U}$ satisfy*

$$\exists R > 0 \ \text{ such that, if } \quad |t| \leq R, \quad \text{then} \quad |f(t) - P(t)| \leq C \sum_{i=1}^{d} |t_i|^{\alpha_i}. \tag{37}$$

The degree condition imposed on P implies uniqueness, as a consequence of the following lemma.

Lemma 28 *Let $\alpha = (\alpha_1, \cdots \alpha_d)$ be d-uple of positive real numbers. If a polynomial P has degree less than α and satisfies*

$$|P(x)| \leq C \sum_{i=1}^{d} |x_i|^{\alpha_i}, \tag{38}$$

for x small enough, then $P = 0$.

Proof: We write $P(x) = \sum C_a x^a$. We consider P in the cone defined by: $\forall i$, $x_i > 0$, where we perform the changes of variable $t_i = x_i^{a_i}$.

Let $\omega_i(a) = a_i/\alpha_i$ and

$$f(t) = \sum_a C_a \prod_i t_i^{\omega_i(a)};$$

then (38) becomes $|f(t)| \leq C|t|$, Writing t in polar coordinates, we get

$$f(t) = \sum_{\omega_a} r^{\omega(a)} g_{\omega(a)}(t),$$

where $\omega(a) = \sum \omega_i(a)$ and the $g_{\omega(a)}$ are homogeneous of degree 0. Ordering the $\omega(a)$ by increasing order, one obtains by induction that each $g_{\omega(a)}$ vanishes, which implies that each term $\sum C_a x^a$ vanishes (where the sum is taken on the indices a such that $\omega(a)$ is given). Since the cone considered has a nonempty interior, this implies that the corresponding polynomial vanishes, and therefore that, forall a, $C_a = 0$.

Let $\alpha = (\alpha_1, \cdots \alpha_d)$ is a d-uple of nonnegative real numbers. We define the *average regularity* $\tilde{\alpha}$ as the harmonic mean of the α_i, i.e.

$$\frac{1}{\tilde{\alpha}} = \frac{1}{d} \sum_{j=1}^{d} \frac{1}{\alpha_j}.$$

The *anisotropy indices* ν_i are the $\nu_i = \tilde{\alpha}/\alpha_i$. Therefore, $\sum \nu_i = d$; note that, in the anisotropic case, $\forall i$, $\nu_i = 1$. The anisotropic pointwise regularity can also be interpreted as a uniform condition inside some elongated ellipsoïds.

Definition 29 *The ε-neighbourhood of x_0 of directions $\mathcal{U}$ and exponent α, denoted by $\mathcal{N}^{\varepsilon}_{\mathcal{U}}$, is the set of points x whose coordinates on the basis $\mathcal{U}$ satisfy*

$$\sum_{i=1}^{d} \left(\frac{t_i}{\varepsilon^{\nu_i}} \right)^2 \leq 1.$$

One easily checks that f belongs to $C^{\alpha}(x_0, \mathcal{U})$ if and only if, for ε small enough,

$$\exists C > 0, \exists P : \quad deg(P) < \alpha \quad \text{and} \quad \sup_{x \in \mathcal{N}^{\varepsilon}_{\mathcal{U},\alpha}} |f(x) - P(x - x_0)| \leq C\varepsilon^{\tilde{\alpha}}. \tag{39}$$

Remarks:

- The usual pointwise Hölder condition corresponds to the case $\alpha_1 = \cdots = \alpha_d = \alpha$.
- If (37) holds, then the one dimensional function $t \to f(x_0 + tu_i)$ belongs to $C^{\alpha_i}(0)$ so that, in each direction u_i, f has Hölder regularity α_i; thus Definition 27 recaptures the intuitive notion we started with.
- In Definition 27, one could start with a nonorthonormal basis $\mathcal{U}$; however, one easily checks that it would not lead to a more general definition of directional smoothness. Indeed, it would be equivalent to the definition we

gave, the corresponding orthonormal basis being obtained from the non-orthonormal one through a Gram-Schmidt orthonormalization procedure, starting with the direction of highest regularity and going down through the vectors attached to smaller and smaller regularity indices.

The definition of directional smoothness supplied by (39) has the additional advantage of supplying a starting point for the definition of the directional smoothness of measures and also for an extension of the $T^p_\alpha(x_0)$ regularity criterium to an anisotropic setting.

- A measure μ belongs to $M^\alpha(x_0,\mathcal{U})$ if and only if, for ε small enough,

$$\exists C > 0, \quad \text{such that} \qquad \mu(\mathcal{N}^\varepsilon_{\mathcal{U},\alpha}) \leq C\varepsilon^{\tilde{\alpha}}. \tag{40}$$

- A function f belongs to $T^p_\alpha(x_0,\mathcal{U})$ if f has slow growth in L^p and if there exist $C > 0$ and a polynomial P of degree less than α such that

$$\forall r \leq R, \quad \left(\frac{1}{\varepsilon^d} \int_{\mathcal{N}^\varepsilon_{\mathcal{U}}} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq C\varepsilon^{\tilde{\alpha}}. \tag{41}$$

Let us now check on a simple example what this notion yields. Let Γ be a smooth curve in $\mathbb{R}^2$. We consider the function

$$f_\alpha(x) = (dist(x, \Gamma))^\alpha .$$

It is intutively clear that, at a point of Γ, f_α has a singularity of order α in the direction orthogonal to Γ, and is smoother in the tangent direction. Indeed, using these directions as basis, one immediately obtains that, at a generic point where the curvature does not vanish,

$$dist(x, \Gamma)^\alpha \leq C(t_1^{2\alpha} + t_2^\alpha)$$

(where t_1 and t_2 are respectively the coordinates on the tangent and normal vectors) and this result is optimal. Therefore, at a generic point of Γ, f has Hölder regularity $(2\alpha, \alpha)$. In this case, the ε-neighbouhoods have parabolic scaling, which gives an additional explanation for the special role played by parabolic scalings, as in the FBI transform, the Hart-Smith transform, or curvelets. Note however that, at inflexion points, f has Hölder regularity $(3\alpha, \alpha)$, and it may even display higher anisotropy, depending on the order of tangency between Γ and its tangent.

The notion of Hölder regularity supplied by Definition 27 does not allow to define directly an Hölder exponent as the "best possible" d-uple $(\alpha_1, \cdots \alpha_d)$

such that f belongs to $C^{\alpha}(x_0,\mathcal{U})$. Indeed this regularity notion does not induce a total order on d-uples, as shown by the example $f(x,y) = |xy|$ whose pointwise (anisotropic) smoothness at 0 is $C^2(0)$. Clearly,

$$\forall \beta > 0, \quad f \in C^{(1,\beta)}(0,E),$$

and f belongs to $C^{(\beta,1)}(0,E)$, where E is the canonical basis of $\mathbb{R}^2$, and the anisotropic smoothness implies that f belongs to $C^{(2,2)}(0,E)$. However these results clearly do not imply an optimal directional smoothness statement encapsulated in one two-variables exponent. The "compatibility relationships" which hold in general between directional regularity conditions are summarized by a partial ordering property:

$$\text{If} \quad \alpha_1 \le \beta_1, \cdots, \alpha_d \le \beta_d, \text{ then} \qquad f \in C^{\beta}(x_0,\mathcal{U}) \Longrightarrow f \in C^{\alpha}(x_0,\mathcal{U}).$$

One can define directional regularity exponents in the following way, which is more coherent with our initial motivation.

Definition 30 *Let $u \in \mathbb{R}^d - \{0\}$. The Hölder exponent of f in the direction u at x_0 is*

$$h_f(x_0,u) = \sup\{\alpha : \exists \varepsilon > 0 \quad f \in C^{(\alpha,\varepsilon,\cdots,\varepsilon)}(x_0,\mathcal{U})\},$$

where $\mathcal{U}$ is an orthonormal basis starting with the vector u.

Using this definition, the function $|xy|$ we already considered has Hölder exponent $+\infty$ along the two coordinate axes, which is natural to expect.

5.2 A criterium of pointwise directional irregularity

We establish a sufficient directional pointwise irregularity condition based on the *anisotropic Gabor-Wavelet transform* (which we will shorten in *aniset transfom*). If $\mathcal{U}$ is an orthonormal basis of $\mathbb{R}^d$, we denote by $\Omega_{\mathcal{U}}$ the linear mapping that maps the canical basis of $\mathbb{R}^d$ to $\mathcal{U}$. Let $\nu = (\nu_1, \cdots, \nu_d)$ be a d-uple satisfying

$$\forall i, \ \nu_i > 0 \quad \text{and} \quad \sum \nu_i = d. \tag{42}$$

We define

$$\phi_{a,\nu}(t) = \frac{1}{a^d}\phi\left(\frac{t_1}{a^{\nu_1}}, \cdots, \frac{t_d}{a^{\nu_d}}\right).$$

Definition 31 *Let f be a locally bounded function with slow growth. Let ϕ : $\mathbb{R}^d \longrightarrow \mathbb{R}$ be a function in the Schwartz class such that $\hat{\phi}(0) = 1$ and $\hat{\phi}$ is supported in the unit ball centered at 0. The aniset transform of f of direction $\mathcal{U}$ and excentricity ν is*

$$d(a, b, \mathcal{U}, \nu, \lambda) = \int_{\mathbb{R}^d} f(x) e^{-i\lambda \cdot x} \phi_{a,\nu} \left(\Omega_{\mathcal{U}}(x - b) \right) dx. \tag{43}$$

The anisotropic wavelets $\phi_{a,\nu} \left(\Omega_{\mathcal{U}}(x - b) \right)$ are called "anisets". The following irregularity criterium extends Proposition 5 in the anisotropic setting.

Theorem 32 *Let f be a locally bounded function with slow growth. If $f \in C^\alpha(x_0, \mathcal{U})$, then there exists $C > 0$ such that, if*

$$\sum_{i=1}^{d} \left(a^{\nu_i} |\lambda_i| \right)^2 \geq 1, \tag{44}$$

and if $a \leq 1$ and $|b - x_0| \leq 1$, then

$$\text{then} \qquad |d(a, b, \mathcal{U}, \nu, \lambda)| \leq C \left(a^{\tilde{\alpha}} + \sum_{i=1}^{d} |(x_0)_i - b_i|^{\alpha_i} \right). \tag{45}$$

Proof of Theorem 32: After performing on f a translation of x_0 and applying the isometry $\Omega_{\mathcal{U}}$, we can assume that $x_0 = 0$ and that $\mathcal{U}$ is the canonical basis of $\mathbb{R}^d$. The Fourier transform of $\phi_{a,\alpha}$ is $\hat{\phi} \left(\xi_1 a^{\nu_1}, \cdots, \xi_d a^{\nu_d} \right)$, therefore, it vanishes in a neighbourhood of λ if (44) holds. Thus, in that case,

$$d(a, b, \mathcal{U}, \nu, \lambda) = \int_{\mathbb{R}^d} (f(x) - P(x)) e^{-i\lambda \cdot x} \phi_{a,\nu} (x - b) \, dx.$$

We split the integral in two terms. The first one corresponds to the ball $|b - x_0| \leq 1$, where we can use (37) (because, since f has slow growth, the constant R in (37) can be chosen arbitraily); it follows that the corresponding integral is bounded by

$$C \int \left(\sum |t_i|^{\alpha_i} \right) \left| \phi \left(\frac{t_1 - b_1}{a^{\nu_1}}, \cdots \frac{t_d - b_d}{a^{\nu_d}} \right) \right| dt.$$

making the change of variable $u_i = (t_i - b_i)/a^{\nu_i}$, we get

$$\begin{aligned} |d(a, b, \mathcal{U}, \nu, \lambda)| &\leq C \int \left(\sum a^{\nu_i \alpha_i} |u_i|^{\alpha_i} + |b_i|^{\alpha_i} \right) |\phi(u_1, \cdots u_d)| dt \\ &\leq CC' \left(a^{\tilde{\alpha}} + \sum |b_i|^{\alpha_i} \right). \end{aligned}$$

The second integral is bounded using the fast decay of ϕ, and the slow growth of f, as in the proof of Proposition 5.

In order to obtain an extension of Theorem 7 in the anisotropic setting, we have to define directional gap sequences.

Definition 33 *Let (λ_n) be a sequence in $\mathbb{R}^d$, let $\mathcal{U}$ be an orthonormal basis of $\mathbb{R}^d$, and let ν be a d-uple satisfying (42). The directional gap θ_n of direction $\mathcal{U}$ and excentricity ν is*

$$\theta_n = \sup\left\{\theta : \ \left(\lambda_n + \mathcal{N}_{\mathcal{U}}^{\theta}\right) \bigcap \{\lambda_m\}_{m\neq n} = \emptyset\right\}.$$

Theorem 34 *Let f be given by (1), where we assume that the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ satisfy (8), and let $x_0 \in \mathbb{R}^d$. If f belongs to $C^\alpha(x_0,\mathcal{U})$, then*

$$\exists C > 0,\ \forall n \in \mathbb{N} \quad \textit{if } \lambda_n \notin \mathcal{N}_{\mathcal{U}}^{\theta_n} \qquad \textit{then} \qquad |a_n| \leq \frac{C}{(\theta_n)^{\tilde{\alpha}}}. \tag{46}$$

Proof of Theorem 34: It is a direct consequence of Theorem 32: We estimate the aniset transform of f at particular points; consider

$$D_m = d\left(\frac{1}{\theta_m}, x_0, \mathcal{U}, \nu, \lambda_m\right). \tag{47}$$

On one hand,

$$\begin{aligned} D_m &= (\theta_m)^d \int \left(\sum_n a_n e^{i(\lambda_n-\lambda_m)\cdot x} \phi_{1/\theta_m,\nu}\left(\Omega_{\mathcal{U}}(x-x_0)\right)\right) dx \\ &= \sum_n a_n \hat{\phi}\left(\frac{(\lambda_m)_1 - (\lambda_n)_1}{(\theta_m)^{\nu_1}} \cdots \frac{(\lambda_m)_d - (\lambda_n)_d}{(\theta_m)^{\nu_d}}\right) e^{i\Omega_{\mathcal{U}}(\lambda_n-\lambda_m)\cdot x_0}, \end{aligned} \tag{48}$$

where $(\lambda_m)_i$ is the i-th component of λ_m on the basis $\mathcal{U}$. Since $\hat{\phi}$ vanishes outside of $B(0,1)$, the definition of θ_n implies that

$$\hat{\phi}\left(\frac{(\lambda_m)_1 - (\lambda_n)_1}{(\theta_m)^{\nu_1}} \cdots \frac{(\lambda_m)_d - (\lambda_n)_d}{(\theta_m)^{\nu_d}}\right) = \delta_{n,m},$$

so that $D_m = a_m$. On the other hand, if $f \in C^\alpha(x_0,\mathcal{U})$, then Theorem 32 implies that, for any m such that $\lambda_n \notin \mathcal{N}_{\mathcal{U}}^{\theta_n}$, $|D_m| \leq C\theta_m^{-\tilde{\alpha}}$; Theorem 34 follows.

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