Comparing Measures of Sparsity

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Abstract—Sparsity of representations of signals has been shown to be a key concept of fundamental importance in fields such as blind source separation, compression, sampling and signal analysis. The aim of this paper is to compare several commonly-used sparsity measures based on intuitive attributes. Intuitively, a sparse representation is one in which a small number of coefficients contain a large proportion of the energy. In this paper, six properties are discussed: (Robin Hood, Scaling, Rising Tide, Cloning, Bill Gates, and Babies), each of which a sparsity measure should have. The main contributions of this paper are the proofs and the associated summary table which classify commonly-used sparsity measures based on whether or not they satisfy these six properties. Only two of these measures satisfy all six: the $p,q$-mean with $p \leq 1, q > 1$ and the Gini Index.

Index Terms—Measures of sparsity, measuring sparsity, sparse distribution, sparse representation, sparsity.

I. INTRODUCTION

Wether with sparsity constraints or with sparsity assumptions, the concept of sparsity is readily used in diverse areas such as oceanic engineering [1], antennas and propagation [2], face recognition [3], image processing [4], [5], and medical imaging [6]. Sparsity has also played a central role in the success of many machine learning algorithms and techniques such as matrix factorization [7], signal recovery/extraction [8], denoising [9], [10], compressed sensing [11], dictionary learning [12], signal representation [13]–[15], support vector machines [16], sampling theory [17], [18], image color restoration [19], and source separation/localization [20], [21]. For example, one method of source separation is to transform the signal to a domain in which it is sparse (e.g., time-frequency or wavelet) where the separation can be performed by a partition of the transformed signal space due to the sparsity of the representation [22], [23]. There has also been research in the uniqueness of sparse solutions in overcomplete representations [24], [25].

There are many measures of sparsity. Intuitively, a sparse representation is one in which a small number of coefficients contain a large proportion of the energy. This interpretation leads to several possible measures. Indeed, there are dozens of measures of sparsity used in the literature. Which of the sparsity measures is the best? In this paper we suggest six desirable characteristics of sparsity used in the literature. Which of the sparsity measures should be understood.

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sixteen measures graphically on data drawn from two sets of parameterized distributions. We select distributions for which we can control the “sparsity.” This allows us to visualize the behavior of the sparsity measures in view of the sparse criteria. In Section V we present some conclusions. The main conclusion is that from the sixteen measures, only the $pq$-mean (with $p \leq 1, q > 1$) and the Gini Index satisfy all six criteria, and, as such, we encourage their use and study.

II. THE SIX CRITERIA

The following are six desirable attributes of a measure of sparsity. The first four, $D_1$ through $D_4$, were originally applied in a financial setting to measure the inequity of wealth distribution in [27]. The last two, $P_1$ and $P_2$, were proposed in [28]. Distribution of wealth can be used interchangeably with distribution of energy of coefficients and where convenient in this paper, we will keep the financial interpretation in the explanations. Inequity of distribution is the same as sparsity. An equitable distribution is one with all coefficients having the same amount of energy, the least sparse distribution.

$D_1$ Robin Hood – Robin Hood decreases sparsity (Dalton’s 1st Law). Stealing from the rich and giving to the poor decreases the inequity of wealth distribution (assuming we do not make the rich poor and the poor rich). This comes directly from the definition of a sparse distribution being one for which most of the energy is contained in only a few of the coefficients.

$D_2$ Scaling – Sparsity is scale invariant (Dalton’s modified 2nd Law [29]). Multiplying wealth by a constant factor does not alter the effective wealth distribution. This means that relative wealth is important, not absolute wealth. Making everyone ten times more wealthy does not affect the effective distribution of wealth. The rich are still just as rich and the poor are still just as poor.

$D_3$ Rising Tide – Adding a constant to each coefficient decreases sparsity (Dalton’s 3rd Law). Give everyone a trillion dollars and the small differences in overall wealth are then negligible so everyone will have effectively the same wealth. This is intuitive as adding a constant energy to each coefficient reduces the relative difference of energy between large and small coefficients. This law assumes that the original distribution contains at least two individuals with different wealth. If all individuals have identical wealth, then by $D_2$ there should be no change to the sparsity for multiplicative or additive constants.

$D_4$ Cloning – Sparsity is invariant under cloning (Dalton’s 4th Law). If there is a twin population with identical wealth distribution, the sparsity of wealth in one population is the same for the combination of the two.

$P_1$ Bill Gates – Bill Gates increases sparsity. As one individual becomes infinitely wealthy, the wealth distribution becomes as sparse as possible.

$P_2$ Babies – Babies increase sparsity. In populations with nonzero total wealth, adding individuals with zero wealth increases the sparseness of the distribution of wealth.

These criteria give rise to the sparsest distribution being one with one individual owning all the wealth and the least sparse being one with everyone having equal wealth.

Dalton [27] proposed that multiplication by a constant should decrease inequality. This was revised to the more desirably property of scale invariance [29]. Dalton’s fourth principle, $D_4$, is somewhat controversial. However, if we have a distribution from which we draw coefficients and measure the sparsity of the coefficients which we have drawn, as we draw more and more coefficients we would expect our measure of sparsity to converge. As stated in [29], $D_4$ captures this concept:

“Mathematically this [D4] requires that the measure of inequality of the population should be a function of the sample distribution function of the population. Most common measures of inequality satisfy this last principle.”

Interestingly, most measures of sparsity do not satisfy this principle, as we shall see.

We define a sparse measure $S$ as a function with the following mapping

$$S : \left( \bigcup_{n \geq 1} \mathbb{C}^n \right) \rightarrow \mathbb{R}$$

(1)

where $n \in \mathbb{N}$ is the number of coefficients. Thus $S$ maps complex vectors to a real number.

There are two crucial, core, underlying attributes which our sparsity measures must satisfy. As all measures satisfy these two conditions trivially we will not comment on them further except to define them.

$A_1$ $S(\tilde{c}) = S(\Pi \tilde{c})$ where $\Pi$ denotes permutation, that is, the sparsity of any permutation of the coefficients is the same. This means that the ordering of the coefficients is not important.

$A_2$ The sparsity of the coefficients is calculated using the magnitudes of the coefficients. This means we can assume we are operating in the positive orthant, without loss of generality.

Using $A_2$ we can rewrite (1) as

$$S : \left( \bigcup_{n \geq 1} \mathbb{R}^+_n \right) \rightarrow \mathbb{R}$$

(2)

which is more consistent with the wealth interpretation.

We will use the convention that $S(\tilde{c})$ increases with increasing sparsity where $\tilde{c} = [c_1 \ c_2 \ \cdots]$ are the coefficient magnitudes. Given vectors

$$\tilde{c} = [c_1 \ c_2 \ \cdots \ c_N]$$

$$\tilde{d} = [d_1 \ d_2 \ \cdots \ d_M]$$

we define concatenation, which we use $|$ to denote, as

$$\tilde{c} | \tilde{d} = [c_1 \ c_2 \ \cdots \ c_N \ d_1 \ d_2 \ \cdots \ d_M].$$

We define the addition of a constant to a vector as the addition of that constant to each element of the vector, that is, for $\alpha \in \mathbb{R}$,

$$\tilde{c} + \alpha = [c_1 + \alpha \ c_2 + \alpha \ \cdots \ c_N + \alpha].$$
We define the multiplication of a vector and a constant as the multiplication of each element of the vector by the constant, that is, for $\alpha \in \mathbb{R}$, 

$$\alpha \vec{c} = [\alpha c_1 \ \alpha c_2 \ \cdots \ \alpha c_N].$$

The six sparse criteria can be formally defined as follows.

**D1 Robin Hood:**

$$S(c_1 \cdots c_i - \alpha \cdots c_j + \alpha \cdots) < S(\vec{c})$$

for all $\alpha, c_i, c_j$ such that $c_i > c_j$ and $0 < \alpha < \frac{c_i - c_j}{2}$.  

**D2 Scaling:**

$$S(\alpha \vec{c}) = S(\vec{c}), \forall \alpha \in \mathbb{R}, \alpha > 0.$$  

**D3 Rising Tide:**

$$S(\alpha + \vec{c}) < S(\vec{c}), \alpha \in \mathbb{R}, \alpha > 0$$

(We exclude the case $c_1 = c_2 = c_3 = \cdots = c_i = \cdots \forall i$ as this is equivalent to scaling.).  

**D4 Cloning:**

$$S(\vec{c}) = S(\vec{d} || \vec{c}) = S(\vec{c} || \vec{d} \cdots || \vec{c}).$$  

**P1 Bill Gates:**

$$S([c_1 \cdots c_i + \beta + \alpha \cdots]) > S([c_1 \cdots c_i + \beta \cdots]).$$  

**P2 Babies:**

$$S(\vec{c} || 0) > S(\vec{c}).$$

As stated above, when proving **D3 Rising Tide** we exclude the scenario where all coefficients are equal. In this case, adding a constant is actually a form of scaling. Another interpretation is that the case with all coefficients equal is, in fact, the minimally sparse scenario and hence adding a constant cannot decrease the sparsity.

**A. Redundancy of P1 and P2**

As one would surmise there is some overlap between the criteria. We present and prove two theorems which demonstrate this overlap. Theorem 2.1 states that if a measure satisfies both criteria D1 and D2, the sparsity measure also satisfies P1. Theorem 2.2 states that a measure satisfying D1, D2 and D4 also satisfies P2.

**Theorem 2.1:** D1 & D2 $\Rightarrow$ P1, that is, if both D1 and D2 are satisfied, P1 is also satisfied.  

**Proof:** Without loss of generality, we begin with the vector $\vec{c}$ ordered in ascending order

$$\vec{c} = [c_1 \ c_2 \ \cdots \ c_N]$$

with $c_1 \leq c_2 \leq \cdots \leq c_N$. We then perform a series of inverse Robin Hood steps to get a vector $\vec{d}$ that is, we take from smaller coefficients and give to the largest coefficient $d_i = c_i - \Delta c_i \ \forall i = 1, 2, \ldots, N - 1$

$$d_N = c_N + \sum_{i=1}^{N-1} \Delta c_i$$

with condition $\Delta < 1$. As these are inverse Robin Hood steps (inverse D1), they increase sparsity and result in the vector

$$\vec{d} = [c_1 - \Delta c_1 \ c_2 - \Delta c_2 \ \cdots \ c_{N-1} - \Delta c_{N-1} \ \Delta c_1 + \cdots + \Delta c_{N-1} + c_N].$$

Without affecting the sparsity we can then scale $(D2) \vec{d}$ by $\frac{1}{1 - \Delta}$ to get

$$\vec{e} = [c_1 \ c_2 \ \cdots \ c_{N-1} \ \frac{1}{1 - \Delta} (\Delta c_1 + \Delta c_2 + \cdots + \Delta c_{N-1} + c_N)]$$

$$= [c_1 \ c_2 \ \cdots \ c_{N-1} \ c_N]$$

where

$$\alpha = \frac{1}{1 - \Delta} (\Delta c_1 + \Delta c_2 + \cdots + \Delta c_N).$$

It is clear that

$$S(\vec{e}) = S(\vec{d}) > S(\vec{c}),$$

which is equivalent to $P1$ with the given $\alpha$ and $\beta = 0$. If we wish to operate on $c_i$ (instead of $c_N$ as above), $\beta$ can be chosen sufficiently large to make the desired coefficient the largest, that is, we set $\beta > c_N - c_i$.  

**Theorem 2.2:** D1, D2 & D4 $\Rightarrow$ P2, that is, if D1, D2 and D4 are satisfied, P2 is also satisfied.  

**Proof:** We begin with vector $\vec{e}$

$$\vec{e} = [c_1 \ c_2 \ \cdots \ c_N].$$

We then clone $(D4)$ this $N + 1$ times to get

$$\vec{\bar{c}} = [\frac{c_1}{N + 1} \ \frac{c_2}{N + 1} \ \cdots \ \frac{c_N}{N + 1}].$$

We then take one of the $\vec{c}$ from $\vec{\bar{c}}$, which we shall refer to as $\vec{\bar{e}}$, and by a series of inverse Robin Hood operations (D1) we distribute this $\vec{\bar{e}}$ in accordance with the size of each element to form new vector $\vec{\bar{d}}$. That is to say, each $c_i$ of each $\vec{c}$ (excluding $\vec{\bar{e}}$) becomes $c_i + \frac{c_N}{N}$ by $N$ consecutive inverse Robin Hood operations which increase sparsity. The result is

$$\vec{\bar{d}} = \begin{bmatrix} \vec{\bar{e}} + \frac{c_N}{N} & \vec{\bar{e}} + \frac{c_N}{N} & \cdots & \vec{\bar{e}} + \frac{c_N}{N} & 0 & 0 & \cdots & 0 \\ \\ \end{bmatrix},$$

$$\vec{\bar{d}} = \begin{bmatrix} \vec{\bar{e}} + \frac{c_N}{N} & \vec{\bar{e}} + \frac{c_N}{N} & \cdots & \vec{\bar{e}} + \frac{c_N}{N} & 0 & 0 & \cdots & 0 \\ \end{bmatrix}.$$  

We can then scale $(D2)$ $\vec{\bar{d}}$ by a factor of $\frac{1}{1 + N}$ without affecting the sparsity to get

$$\vec{\bar{e}} = \begin{bmatrix} \vec{\bar{e}} & \vec{\bar{e}} & \cdots & \vec{\bar{e}} & 0 & 0 & \cdots & 0 \\ \end{bmatrix}.$$
TABLE I
COMMONLY USED SPARSITY MEASURES MODIFIED TO BECOME MORE
POSITIVE FOR INCREASING SPARSITY

<table>
<thead>
<tr>
<th>Measure</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell^0$</td>
<td>$# {j : c_j = 0}$</td>
</tr>
<tr>
<td>$\ell^1$</td>
<td>$# {j : c_j \leq \epsilon}$</td>
</tr>
<tr>
<td>$-\ell^1$</td>
<td>$-\sum_j c_j$</td>
</tr>
<tr>
<td>$-\ell^p$</td>
<td>$-\left(\sum_j c_j^p\right)^{1/p}$, $0 &lt; p &lt; 1$</td>
</tr>
<tr>
<td>$\ell^2$</td>
<td>$\sum_j c_j^2$</td>
</tr>
<tr>
<td>$\ell^\infty$</td>
<td>$\max_j</td>
</tr>
<tr>
<td>$-\tanh_{a,b}$</td>
<td>$-\sum_j \tanh\left(ac_j\right)^b$</td>
</tr>
<tr>
<td>$-\log$</td>
<td>$-\sum_j \log(1 + c_j^2)$</td>
</tr>
<tr>
<td>$\kappa_4$</td>
<td>$\sum_j c_j^4 / \left(\sum_j c_j^2\right)^2$</td>
</tr>
<tr>
<td>$u_\theta$</td>
<td>$1 - \min_{i=1,2,\ldots,N-\lfloor \theta N \rfloor + 1} \sum_{j \leq \lfloor \theta N \rfloor} c_j / \sum_{j \leq \lfloor \theta N \rfloor} c_j$ s.t. $\lfloor \theta N \rfloor \neq N$ for ordered data, $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(N)}$, $-\ell^p$</td>
</tr>
<tr>
<td>$H_S$</td>
<td>$-\sum_j \log c_j^2$ where $\tilde{c}_j = \frac{c_j^2}{\left</td>
</tr>
<tr>
<td>$H'_S$</td>
<td>$-\sum_j \log c_j^2$</td>
</tr>
<tr>
<td>Hoyer</td>
<td>$(\sqrt{N} - \sum_j c_j / \sqrt{\sum_j c_j^2}) / (\sqrt{N} - 1)^{-1}$</td>
</tr>
<tr>
<td>$pq$-mean</td>
<td>$\left(\frac{1}{N} \sum_{k=1}^N \frac{c_k}{\left</td>
</tr>
</tbody>
</table>

which by cloning (D4) we know is equivalent to

$$\tilde{F} = \begin{bmatrix} \tilde{c} & 0 \end{bmatrix}.$$

In summation, we have shown that

$$S(\tilde{c}) = S(\tilde{c}) < S(\tilde{D}) = S(\tilde{F}) = S(\tilde{F})$$

that is,

$$S(\tilde{c}) < S(\tilde{c}|0)$$

which is also known as $P2$.

III. THE MEASURES OF SPARSITY

In this section we discuss a number of popular sparsity measures. These measures are used to calculate a number which describes the sparsity of a vector $\tilde{c} = [c_1, c_2, \ldots, c_N]$. The measures’ monikers and their definitions are listed in Table I. Some measures in Table I have been manipulated (in general negated) to ensure that an increase in sparsity results in a positive increase in the sparse measure.

In [30], the $\ell^0$, $\ell^1$, $\ell^p$, $\tanh_{a,b}$, log and $\kappa_4$ were compared. The most commonly used and studied sparsity measures are the $\ell^p$ norm-like measures

$$||\tilde{c}||_p = \left(\sum_j c_j^p\right)^{1/p} \text{ for } 0 \leq p \leq 1,$$

The $\ell^0$ measure simply calculates the number of non-zero coefficients in $\tilde{c}$,

$$||\tilde{c}||_0 = \#\{j \neq 0, j = 1, \ldots, N\}.$$

The $\ell^0$ measure is the traditional sparsity measure in many mathematical settings. However, it is unsuited to many practical scenarios, for two reasons: its derivative contains no information and its poor behavior in the presence of noise. The derivative of the measure is always zero when defined and as such the $\ell^0$ is not commonly used in optimization problems where sparsity is the desired outcome. Exhaustive search is the only method of finding the sparsest solution when using the $\ell^0$ measure and thus approximations are usually used [31], [32]. Additionally, the presence of noise makes the $\ell^0$ measure completely inappropriate. In noisy settings, the $\ell^0$ measure is sometimes modified to $\ell^p$ where we are interested in the number of coefficients that are greater than a threshold $\epsilon$ [33]. Clearly, the value of $\epsilon$ is crucial for $\ell^p$ to be meaningful. This dependency on $\epsilon$ is undesirable. As optimization using $\ell^0$ is difficult because the gradient also yields no information, $\ell^p$ with $0 < p < 1$ is often used in its place, [34]. The $\ell^1$ measure, that is, $\ell^p$ with $p = 1$, approximates the $\ell^0$ measure and is easily calculated. Under this measure, large coefficients are considered more important than small coefficients unlike the $\ell^0$ measure. In some settings, the $\ell^1$ solution can be used to find the support of the $\ell^0$ solution [35]. The $\ell^0$ solution can be found efficiently via linear programming and is thus used in many optimization problems [36], [37].

In [30] several alternative measures of sparsity are noted which approximate the $\ell^0$ measure but emphasize different properties. $\tanh_{a,b}$ is sometimes used in place of $\ell^p$, $0 < p < 1$, as it is limited to the range $(0,1)$ and better models $\ell^0$ and $\ell^p$ in this respect. A representation is more sparse if it has one large component, rather than dividing up the large component into two smaller ones. $\tanh_{a,b}$ and $\ell^p$ preserve this. In [28] it is shown that the log measure enforces sparsity outside some range, but for distributions with low energy coefficients the opposite is achieved by effectively spreading the energy of the small components. $\kappa_4$ is the kurtosis which measures the peakedness of a distribution [38]. $u_\theta$ measures the smallest range which contains a certain percentage of the data. This is achieved by sorting the data and determining the minimum difference between the largest and smallest sample in a range containing the specified percentage ($\theta$) of data points as a fraction of the total range of the data. The reason that a continuous parameter $\theta$ is used in the model is to maintain compatibility with pre-existing literature.

For measuring “diversity,” [39], [40] use some different measures. Three of these are entropy measures: the Shannon entropy diversity measure $H_S$, a modified version of the Shannon entropy diversity measure $H'_{S}$ and the Gaussian entropy diversity measure $H_{G}$. They also extend the $\ell^p$ measure to negative exponents, that is, $-1 < p < 0$. We call this measure $\ell^p_-$ to avoid confusion.

For some measures, a normalization of the measure can be employed to satisfy more constraints. For example, if we define kurtosis in a different way and denote it $\hat{\kappa}_4$:

$$\hat{\kappa}_4 = \frac{E_N\{x^4\}}{(E_N\{x^2\})^2}.$$
where \( E_N \) denotes empirical mean,
\[
E_N \{ z \} = \frac{1}{N} \sum_{i=1}^{N} z_i,
\]
we obtain a scaled version of the previously defined kurtosis, \( \kappa_4 \)
\[
\tilde{\kappa}_4 = \frac{N \sum_{k=1}^{N} x^4}{(\sum_{k=1}^{N} x^2)^2} = N \kappa_4.
\]
This normalized kurtosis (\( \tilde{\kappa}_4 \)) satisfies two more criteria than
the unscaled kurtosis \( \kappa_4 \). Another example of this is the Hoyer
measure [41] which is a normalized version of the \( l_p^2 \) measure
as is obvious from its definition, \( (\sqrt{N} - \frac{\sqrt{p}}{p}) (\sqrt{N} - 1)^{-1} \). We
discuss the normalization of other measures in Section IV-B.

The \( pq \)-mean represents an interesting general family of mea-

sures and is the ratio of the generalized \( p \) mean and the gener-
alized \( q \) mean. This measure appears in [42] with \( q = 2 \) and
\( 0 < p \leq 1 \). The normalized kurtosis \( \kappa_4 \) is the \( pq \)-mean with
\( p = 2 \) and \( q = 4 \) and the \( \frac{\kappa_4}{\sqrt{N}} \) is a scaled version of the \( pq \)-mean
with \( p = 1 \) and \( q = 2 \).

A. The Gini Index

Given a vector, \( \vec{c} = [c_1, c_2, c_3, \ldots] \), we order from
smallest to largest, \( c_{(1)} \leq c_{(2)} \leq \ldots \leq c_{(N)} \) where
(1), (2), \ldots, (N) are the new indices after the sorting op-
eration. The Gini Index is given by
\[
S(\vec{c}) = 1 - 2 \sum_{k=1}^{N} \frac{c_{(k)}}{[k][N]} \left( \frac{N-k + \frac{1}{2}}{N} \right).
\]  

The Gini Index also has an interesting graphical interpreta-
tion which we see in Fig. 1. If percentage of coefficients versus
percentage of total coefficient value is plotted for the sorted co-
efficients we can define the Gini Index as twice the area between
this line and the 45\(^\circ\) line. The 45\(^\circ\) line represents the least sparse
distribution, that with all the coefficients being equal.

If we have a distribution from which we draw coefficients and
measure the sparsity of the coefficients which we have drawn, as we draw more and more coefficients we would expect our
measure of sparsity to converge. The Gini Index meets these
expectations. The Gini Index of a distribution with probability
density function \( f(x) \) (which satisfies \( f(x) = 0, x < 0 \) and cumula-
tive distribution function \( F(x) \) is given by
\[
G = 1 - 2 \int_0^1 \frac{\int_0^x f(t) dt}{\int_0^\infty f(t) dt} dF(x).
\]

As a side note, the Gini Index was originally proposed in eco-
nomics as measure of the inequality of wealth [27], [29], [43],
[44] and is still studied in relation to wealth distribution as well
as other areas. [45]–[48] “Inequality in wealth” in signal proc-
ressing language is “efficiency of representation” or “sparsity.”
The utility of the Gini Index as a measure of sparsity has been
demonstrated in [28], [49]–[51].

Fig. 1. Percentage of coefficients versus percentage of total coefficient value is plotted for the sorted coefficients for \([0, 0.0.0,0.1]\) (top) and \([1.1,2.3,10]\) (bottom). The Gini Index is twice the shaded area.

IV. COMPARISON OF SPARSITY MEASURES

In this section we present the main result of the paper, the
comparison of the measures using the criteria. Many of the mea-
sures fail for simple test cases which prove non-compliance.
For example, \([0,1,3,5]\) is more sparse than \([0,2,3,4]\) because a Robin Hood operation maps one sequence to the other. Six of
the measures do not correctly handle this case. Others fail on
similar examples. Seven of the measures, however, satisfy \( D1 \).

An example for each sparse criterion is given in Table II along
with the desired outcome when the sparsity of the examples are
measured with sparsity measure \( S(\cdot) \). Table III details which
of the six sparse criteria hold for each of the sixteen measures.
The information is based on proofs and counter-examples which are
contained in their entirety in Appendices A and B. There are es-
sentially two types of proof, Type A and Type B. Type A is the
standard form of proof which uses inequalities, an example of
which is the following:

\[ \text{Theorem 4.1:} \frac{\mu}{\mu} \text{ satisfies } D1 \]
\[ S([c_1 \ldots c_j - \alpha \ldots c_j + \alpha \ldots]) < S(\vec{c}) \]
such that we can restate the above as

\[ \text{satisfies} \quad \text{and the Hoyer measure satisfy most of the criteria.} \]

Expanding this we obtain

\[ \frac{\partial}{\partial \alpha} \left[ - \left( \sum_{i \neq i, j} c_i^p + (c_i - \alpha)^p + (c_j + \alpha)^p \right) \right] < 0. \]

which holds true if

\[ (c_j + \alpha)^{p-1} - (c_i - \alpha)^{p-1} > 0. \]

As \( p < 1 \) we can rewrite the above as

\[ \frac{1}{c_j + \alpha} - \frac{1}{c_i - \alpha} > 0 \]

which is necessarily true as it is one of the constraints upon \( \alpha \).

From Table III we can see that \( D3 \) (Rising Tide) is satisfied by most measures. Perhaps this shows that relative size of coefficients is of the utmost importance when desiring sparsity. As previously mentioned, most measures do not satisfy \( D4 \) (Cloning). Each of the other criteria is satisfied by a varying number of the sixteen measures of sparsity. This demonstrates the variety of attributes to which measures of sparsity attach importance. \( \kappa_4 \) and the Hoyer measure satisfy most of the criteria.

The Gini Index and the \( pq \)-mean are measures of the number of the sixteen measures of sparsity. This demonstrates the variety of attributes to which measures of sparsity attach importance.

**A. Normalization of the Sparsity Measures**

In this section we examine the effects of a simple normalization of the sparsity measures. We have already discussed two normalizations—normalized kurtosis (\( \kappa_4 \)) and the Hoyer measure which is a normalized version of \( \frac{E}{p^2} \). From Table III we can see that the Hoyer measure satisfies more criteria than \( \frac{E}{p^2} \). Similarly, the normalized kurtosis \( \kappa_4 \) satisfies more constraints than \( \kappa_4 \).

By normalizing a measure to the length of the vector being measured we can affect how the measure responds to criteria \( D4 \)

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**TABLE II**

**Most Common Counter-Example for a Given Property With Measure of Sparsity and Desired Outcome With Sparsity Measure \( S(\cdot) \)**

<table>
<thead>
<tr>
<th>Property</th>
<th>Most common counter-example</th>
<th>Desired outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D1 )</td>
<td>([0, 1, 3, 5]) vs ([0, 2, 3, 4])</td>
<td>( S([0, 1, 3, 5]) &gt; S([0, 2, 3, 4]) )</td>
</tr>
<tr>
<td>( D2 )</td>
<td>([0, 1, 3, 5]) vs ([0, 2, 6, 10])</td>
<td>( S([0, 1, 3, 5]) = S([0, 2, 6, 10]) )</td>
</tr>
<tr>
<td>( D3 )</td>
<td>([0, 1, 3, 5]) vs ([0, 1, 3, 5])</td>
<td>( S([0, 1, 3, 5]) &lt; S([1, 5, 3, 5]) )</td>
</tr>
<tr>
<td>( D4 )</td>
<td>([0, 1, 3, 5]) vs ([0, 1, 0, 1, 3, 5])</td>
<td>( S([0, 1, 3, 5]) = S([0, 0, 0, 1, 3, 5]) )</td>
</tr>
<tr>
<td>( P1 )</td>
<td>([0, \beta]) vs ([0, (\alpha + \beta)])</td>
<td>( S([0, \beta]) &lt; S([0, (\alpha + \beta)]) )</td>
</tr>
<tr>
<td>( P2 )</td>
<td>([0, 1, 3, 5]) vs ([0, 0, 0, 1, 3, 5])</td>
<td>( S([0, 1, 3, 5]) &lt; S([0, 0, 0, 1, 3, 5]) )</td>
</tr>
</tbody>
</table>

---

**TABLE III**

**Comparison of Different Sparsity Measures Using Criteria Defined in Section II**

<table>
<thead>
<tr>
<th>Measure</th>
<th>( D1 )</th>
<th>( D2 )</th>
<th>( D3 )</th>
<th>( D4 )</th>
<th>( P1 )</th>
<th>( P2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^p )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( \mu^p )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( \ell^p )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( \kappa_4 )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( u_g )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( H_G )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
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<td>( H_C )</td>
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<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>Hoyer</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>( pq )-mean</td>
<td>( p &lt; 1, q &gt; 1 )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
<tr>
<td>Gini</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
<td>( \checkmark )</td>
</tr>
</tbody>
</table>

for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2} \).

**Proof:** As \( \ell^p = \sqrt[p]{\sum_{i \neq i, j} c_i^p + (c_i - \alpha)^p + (c_j + \alpha)^p} \) we can restate the above as

\[ \sqrt[p]{\sum_{k \neq k, j} c_k^p + (c_i - \alpha)^p + (c_j + \alpha)^p} < \sqrt[p]{\sum_{k \neq k} c_k^p}. \]

This simplifies to

\[ (c_i - \alpha)^p + (c_j + \alpha)^p < (c_i - \alpha)^p + (c_j + \alpha)^p. \]

Expanding this we obtain

\[ c_i^p - 2c_i \alpha + \alpha^2 + 2c_j \alpha + \alpha^2 < c_i^p + c_j^p, \]

which we know is true as \( 0 < \alpha < \frac{c_i - c_j}{2} \).

A Type B proof on the other hand uses derivatives, for example.

**Theorem 2.4:** \( -\ell^p \) satisfies \( D1 \)

\[ S([c_1, \cdots, c_i - \alpha, \cdots, c_j + \alpha, \cdots]) < S(\ell) \]

for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2} \).

**Proof:**

\[ -\ell^p = -\left( \sum_{k} c_k^p \right)^{1/p}, \quad 0 < p < 1. \]
TABLE IV
COMPARISON OF SPARSITY MEASURES NORMALIZED TO THE LENGTH OF THE VECTOR (DENOTED BY N) USING CRITERIA DEFINED IN SECTION II. THE MEASURES ARE, IN GENERAL, DIVIDED BY THE LENGTH OF THE VECTOR BEING MEASURED. ADDITIONAL CONSTRAINTS WHICH ARE SATISFIED AS A RESULT OF THE NORMALIZATION ARE MARKED WITH ●. NOTE THAT K_d IS NORMALIZED IN A DIFFERENT MANNER TO THE OTHER MEASURES

<table>
<thead>
<tr>
<th>Measure</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^2_f$</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^2_p$</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$r^2_{\bar{N}}$</td>
<td></td>
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<tr>
<td>$r^2_{\bar{N}^2}$</td>
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</tr>
<tr>
<td>$r^2_{\bar{N}^3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^2_{\hat{N}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^2_{\hat{N}^2}$</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$r^2_{\hat{N}^3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hoyer</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pG-mean</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gini</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The other criteria compare vectors of the same length and are thus not affected by a normalization/scaling which is a function of the vector length. In Table IV we show the additional criteria satisfied by measures normalized to the length of the vector, where $N$ is the length of the vector being measured. Not all measures are scaled in this way as some are already normalized in some fashion, for example, the Hoyer measure is a normalized version of $\frac{r^2_f}{P_f}$. We omit the proofs of the normalized measures as they are trivial extensions of the proofs of the previously defined measures.

B. Numerical Sparse Analysis

In this section we present the results of using the sixteen sparse measures to measure the sparsity of data drawn from a set of parameterized distributions. By applying the sixteen measures to data drawn from these distributions, we can visualize the criteria. The examples are based on the premise that all coefficients being equal is the least sparse scenario and all coefficients being zero except one is the most sparse scenario. In the first experiment we draw a variable number of coefficients from a probability distribution and measure their sparsity. We expect sets of coefficients from the same distribution to have a similar sparsity. As we increase the number of coefficients we expect the measure of sparsity to converge. In this experiment we examine the sparsity of coefficients from a Poisson distribution with parameter $\lambda = 5$ as a function of set size. From the sparsity plot in Fig. 2 we can see that four measures converge. They are $K_d$, the Hoyer measure, the $pG$-mean ($p = 1, q = 3$) and the Gini Index. As this experiment is similar in nature to $D_4$ we expect the Gini Index to converge. We also expect that for any $p \leq 1, q > 1$ the $pG$-mean will converge and include one example ($p = 1, q = 3$). The convergence of Hoyer measure is somewhat unsurprising because for large $N$, that is, for sufficiently long vectors, the Hoyer measure approximately satisfies $D_4$. By this we mean that if the vector $\mathbf{v}$ is cloned $M$ times

$$S(\mathbf{v}, \ldots, \mathbf{v}) = \left(\frac{\sqrt{N} - 1}{\sqrt{N} - \frac{1}{\sqrt{M}}}\right)^M S(\mathbf{v})$$

and clearly

$$\lim_{N \to \infty} \left(\frac{\sqrt{N} - 1}{\sqrt{N} - \frac{1}{\sqrt{M}}}\right) = 1$$

which means that in the limit the Hoyer measure satisfies $D_1$. The results presented in the figure are scaled for clearer visualization so that the sparsity falls between 0 and 1.

In the second experiment we take coefficients from a Bernoulli distribution where coefficients are either 0 with probability $p$ or 1 with probability $1 - p$. For this experiment the set size remains constant and the probability $p$ varies from 0 to 1. With a low $p$ most coefficients will be 1 and very few zero. The energy distribution of such a set is not sparse and accordingly has a low value (see Fig. 3). As $p$ increases so should the sparsity measure. We can see this is the case in some form for all of the measures except $H'_S$. We note that $K_d$ does not rise steadily with increasing $p$ but rises dramatically as the set approaches its sparsest. This is of some concern if optimizing sparsity using $K_d$ and there is not much indication that the distribution is getting more sparse until its already quite sparse. The $pG$-mean ($p = 1, q = 3$) shows similar behavior which is unsurprising due to the fact that $K_d$ is the $pG$-mean for $p = 2, q = 4$.

V. Conclusion

In this paper we have presented six intuitive attributes of a sparsity measure. Having defined these attributes mathematically, we then compared commonly-used measures of sparsity. The goal of this paper is to provide motivation for selecting a particular measure of sparsity. Each measure emphasizes different combinations of attributes and this should be addressed when selecting a sparsity measure for an application. We can see from the main contribution of this paper, Table III and the associated proofs in Appendices A and B, that the only measures to satisfy all six criteria are the $pG$-mean ($p \leq 1, q > 1$) and the Gini Index. This result for the Gini Index aligns well with [49] in which it is shown that the Gini Index is an indicator for when sources are separable, a property which itself relies on sparsity. The Hoyer measure [41] and the normalized kurtosis ($K_d$) satisfy five of the six criteria. The Hoyer measure fails only $D_4$ (Cloning), which is, admittedly an arguable criterion for certain applications. For applications in which the number of coefficients is fixed the Hoyer measure satisfy all criteria. However, the normalized kurtosis ($K_d$) fails $D_1$ (Robin Hood) which is arguably the most crucial of all the measures.

We have also presented two graphical examples of the performance of the measures when quantifying the sparsity of samples from a distribution. Again, both the Gini Index and the Hoyer measure outperform the other measures, illustrating their utility.
Sparsity is used in many applications but with few exceptions it is not studied as a concept in itself. We hope that this work will not just encourage the use of the Gini Index and \( pq \)-mean (\( p \leq 1, q > 1 \)) but encourage users of sparsity to consider in more depth the concept of sparsity.

APPENDIX

We use these measures to calculate a number which describes the sparsity of a set of coefficients \( \mathbf{c} = [c_1, c_2, \ldots, c_N]\).

We ignore the trivial cases, for example, \( \omega_2 \) with \( \alpha = 1 \).

D1 Robin Hood:

\[
S([c_1 \cdots c_i - \alpha \cdots c_j + \alpha \cdots]) < S(\mathbf{c})\quad \text{for all }\quad c_i > c_j \quad \text{and } 0 < \alpha < \frac{c_i - c_j}{2}.
\]

D2 Scaling:

\[
S(\alpha \mathbf{c}) = S(\mathbf{c}), \quad \forall \alpha \in \mathbb{R}, \quad \alpha > 0.
\]

D3 Rising Tide:

\[
S(\alpha + \mathbf{c}) < S(\mathbf{c}), \quad \alpha \in \mathbb{R}, \quad \alpha > 0 \quad \text{(We exclude the case }\quad c_1 = c_2 = c_3 = \cdots = c_i = \cdots \forall_i \text{ as this is equivalent to scaling.)}.
\]

D4 Cloning:

\[
S(\mathbf{c}) = S(\mathbf{c}|\mathbf{c}) = S(\mathbf{c}|\mathbf{c}|\mathbf{c}) = S(\mathbf{c}|\mathbf{c}|\cdots)\|\mathbf{c}.
\]

P1 Bill Gates:

\[
\forall i, \exists \beta = \beta_i > 0, \text{ such that } \forall \alpha > 0:
\]

\[
S([c_1 \cdots c_i + \beta + \alpha \cdots ]) > S([c_1 \cdots c_i + \beta \cdots]).
\]

P2 Babies:

\[
S(\mathbf{c}|0) > S(\mathbf{c}).
\]

A. Counter Examples

The most parsimonious method of showing noncompliance with the sparse criteria is through the following simple counter-examples. As an example we take the \( -\ell^3 \) measure and D1. D1 states that the \( -\ell^3 \) measure of \([0,1,3,5]\) should be greater than the \( -\ell^1 \) measure of \([0,2,3,4]\). However, for \( -\ell^1 \):

\[
S([0,1,3,5]) = -9 \\
S([0,2,3,4]) = -9.
\]

As the Robin Hood operation had no effect on the sparsity of the vectors as measured by the \( \ell^1 \) measure the measure does not satisfy D1.

Counter Example A.1:

\([0,1,3,5] \) versus \([0,2,3,4]\).

Counter Example A.1 (*):

\([3,1,2] \) versus \([31,99,2]\).

Counter Example A.2:

\([0,1,3,5] \) versus \([0,2,6,10]\).

Counter Example A.3:

\([1,3,5] \) versus \([1,5,3,5,5,5]\).
Fig. 3. Sparsity vs $p$ for a Bernoulli distribution with coefficients being 0 with probability $p$ and 1, otherwise. The measures are scaled to fit between a sparsity range of 0 to 1.

**Counter Example A.3 (⋆):**


**Counter Example A.4:**

$[0, 1, 3, 5]$ versus $[0, 0, 1, 1, 3, 5]$.

**Counter Example A.5:**

$[0, \beta]$ versus $[0, (\alpha + \beta)]$.

**Counter Example A.6:**

$[0, 1, 3, 5]$ versus $[0, 0, 0, 1, 3, 5]$.

### B. Proofs

This section contains the proofs that were longer than Table V permitted. The obvious method of proving that the measures satisfy the criteria is to plug the formulae for the measures into the mathematical definitions of the six criteria. Another method used below is to differentiate the modified sparse measure with respect to the parameter that modifies it and observe the result. For example, if we show that $\frac{\partial S[\alpha + \mathcal{E}]}{\partial \alpha} < 0$ for $\alpha > 0$ this proves $D3$ as any change in $\alpha$ causes the measure to drop.

1) $-\ell^p$ and $D1$:

**Theorem A.1:** $-\ell^p$ satisfies $D1$

$S(c_1 \cdots c_i - \alpha \cdots c_j + \alpha \cdots) < S(\mathcal{E})$

for all $\alpha, c_i, c_j$ such that $c_i > c_j$ and $0 < \alpha < \frac{c_i - c_j}{2}$.

**Proof:**

$-\ell^p = -\left( \sum_{k} c_k^p \right)^{1/p}, \ 0 < p < 1.$

We wish to show that the following holds true for all $\alpha, c_i, c_j$ such that $c_i > c_j$ and $0 < \alpha < \frac{c_i - c_j}{2}$

$$\frac{\partial}{\partial \alpha} \left[-\left( \sum_{k \neq i, j} c_k^p + (c_i - \alpha)^p + (c_j + \alpha)^p \right)^{1/p} \right] < 0$$

$$= \frac{1}{p} \left( \sum_{k \neq i, j} c_k^p + (c_i - \alpha)^p + (c_j + \alpha)^p \right)^{1/p-1} \times \left( -p(c_i - \alpha)^{p-1} + p(c_j + \alpha)^{p-1} \right) < 0.$$ 

Which holds true if

$$(c_j + \alpha)^{p-1} - (c_i - \alpha)^{p-1} > 0.$$ 

As $p - 1 < 0$ we can rewrite the above as

$$\frac{1}{(c_j + \alpha)^{1-p}} - \frac{1}{(c_i - \alpha)^{1-p}} > 0$$

$$\frac{1}{(c_j + \alpha)} > \frac{1}{(c_i - \alpha)}$$

$$c_i - \alpha > c_j + \alpha$$

$$c_i - c_j > 2\alpha.$$
TABLE V
GUIDE TO COUNTER-EXAMPLES AND PROOFS EACH FOLLOWED BY REFERENCE NUMBER. A √ indicates compliance of the measure with the relevant criterion. "obv" means that the proof is obvious and as such is not included. (1 p ≤ 1, q > 1)

<table>
<thead>
<tr>
<th>Measure</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$P_1$</th>
<th>$P_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^p$</td>
<td>C.Ex A.1</td>
<td>√</td>
<td>obv</td>
<td>C.Ex A.3</td>
<td>C.Ex A.4</td>
<td>C.Ex A.5</td>
</tr>
<tr>
<td>$\rho^q$</td>
<td>C.Ex A.1</td>
<td>C.Ex A.2</td>
<td></td>
<td>C.Ex A.4</td>
<td>C.Ex A.5</td>
<td>√</td>
</tr>
<tr>
<td>$-\rho^p$</td>
<td>C.Ex A.1</td>
<td>C.Ex A.2</td>
<td>√</td>
<td>obv</td>
<td>C.Ex A.4</td>
<td>C.Ex A.5</td>
</tr>
<tr>
<td>$-\rho^q$</td>
<td>√ Proof B1</td>
<td>C.Ex A.2</td>
<td>√</td>
<td>Proof B2</td>
<td>C.Ex A.4</td>
<td>C.Ex A.5</td>
</tr>
<tr>
<td>$\rho^p$</td>
<td>√ Proof B3</td>
<td></td>
<td>obv</td>
<td>Proof B4</td>
<td>C.Ex A.4</td>
<td>√</td>
</tr>
<tr>
<td>$-\rho^q$</td>
<td>√ Proof B6</td>
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<td></td>
<td>obv</td>
<td>Proof B7</td>
<td>C.Ex A.4</td>
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<td>√</td>
<td>Proof B8</td>
<td>C.Ex A.4</td>
<td>C.Ex A.5</td>
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<tr>
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<td>C.Ex A.1 (*)</td>
<td>Proof B9</td>
<td>√</td>
<td>Proof B10</td>
<td>C.Ex A.4</td>
<td>√</td>
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<td>Proof B13</td>
<td>√</td>
<td>obv</td>
<td>Proof B14</td>
<td>√ Proof B15</td>
<td>√ Proof B16</td>
</tr>
<tr>
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<td>C.Ex A.2</td>
<td>C.Ex A.3(*)</td>
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<td>√ Proof B12</td>
<td>C.Ex A.6</td>
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<tr>
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<td>C.Ex A.4</td>
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</tr>
<tr>
<td>$H_{G^*}$</td>
<td>C.Ex A.1</td>
<td>C.Ex A.2</td>
<td>C.Ex A.3(*)</td>
<td>C.Ex A.4</td>
<td>C.Ex A.5</td>
<td>C.Ex A.6</td>
</tr>
<tr>
<td>$H^*_G$</td>
<td>C.Ex A.1</td>
<td>C.Ex A.2</td>
<td>C.Ex A.3(*)</td>
<td>C.Ex A.4</td>
<td>C.Ex A.5</td>
<td>C.Ex A.6</td>
</tr>
<tr>
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<td>√ Proof B19</td>
<td></td>
<td>obv</td>
<td>Proof B20</td>
<td>C.Ex A.4</td>
<td>√ Proof B21</td>
</tr>
</tbody>
</table>

which is necessarily true as it is one of the constraints upon $\alpha$. which we know is true as $0 < \alpha < \frac{c_{i-m}}{2}$.

4) $\frac{C}{F}$ and $D_3$:

Theorem A.4: $\frac{C}{F}$ does not satisfy $D_3$

$S(\alpha + \bar{\alpha}) < S(\bar{\alpha})$, $\alpha \in \mathbb{R}$, $\alpha > 0$.

Proof:

$$\sqrt[1/p]{\frac{\sum_{j}c_j^{2} + c_{j} + \alpha + ...}{\sum_{k}c_k^{p}}} < \sqrt{\frac{\sum_{j}c_j^{2} + (\alpha + c_{j} + \alpha + ...)}{\sum_{k}c_k^{p}}}$$

To simplify matters we make the following substitutions

$$s_1 = \sum_j c_j$$
$$s_2 = \sum_{j}c_j^2$$

(6)

and note that $s_1^2 > s_2$. We now have

$$\sqrt{s_2 + 2\alpha s_1 + N\alpha^2} < \sqrt{s_2}$$

$$s_1^2(s_2 + 2\alpha s_1 + N\alpha^2) < s_2(s_1^2 + 2s_1N\alpha + N^2\alpha^2)$$

$$\alpha < \frac{N}{2s_1} \left( \frac{s_2 - s_1^2}{Ns_1^2 - s_2} \right),$$

which is false as $\left( \frac{s_2 - s_1^2}{Ns_1^2 - s_2} \right) < 0$ which violates the condition $\alpha > 0$.

5) $\frac{C}{F}$ and $P_1$:

Theorem A.5: $\frac{C}{F}$ satisfies $P_1$

$\forall i, \exists \beta = \beta_i > 0$, $\forall \alpha > 0$:

$$S([c_1 \ldots c_i + \beta + \alpha + \ldots]) > S([c_1 \ldots c_i + \beta \ldots])$$
Proof: We make the following substitutions
\[ s_1 = \sum_j c_j \]
\[ s_2 = \sum_j c_j^2 \]
and wish to show that
\[ \frac{\sqrt{s_2 + \alpha^2 + 2(\alpha + \alpha \beta + \alpha c_1 + \beta c_1) - \alpha^2}}{\sqrt{s_2 + \beta^2 + 2\beta c_1}} > \frac{\sqrt{s_2 + \beta^2 + 2\alpha \beta c_1}}{\sqrt{s_2 + \alpha^2 + 2\alpha c_1}}. \]

Squaring both sides and cross-multiplying gives
\[ \alpha > \frac{2s_1 s_2 + 2\beta^2 c_1 - 2\beta s_1^2 - 2\alpha c_1 s_1^2}{s_1^2 + 2s_1 \beta - s_2 - 2\beta c_1}. \]

We want \( RHS < 0 \) and therefore want a \( \beta \) such that
\[ \frac{2s_1 s_2 + 2\beta^2 c_1 - 2\beta s_1^2 - 2\alpha c_1 s_1^2}{s_1^2 + 2s_1 \beta - s_2 - 2\beta c_1} \leq 0. \]

As the denominator is always positive, we are only interested in the numerator, that is, finding a \( \beta \) such that
\[ s_1 s_2 + \beta^2 c_1 - \beta s_1^2 - c_1 s_1^2 \leq 0. \]

This is satisfied for \( \beta = s_1 \)
\[ s_1 s_2 + s_1^2 c_1 - s_1^3 - c_1 s_1^2 \leq 0, \]
which is clearly true.

6) \( -\tanh_{a,b} \) and D1:
Theorem A.6: \( -\tanh_{a,b} \) satisfies D1
\[ S([c_1 \cdots c_i - \alpha \cdots c_j + \alpha \cdots]) < S(\vec{c}), \]
for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2}. \)
Proof: Need to show that
\[ -\tanh(ac_i - \alpha c_i) - \tanh(ac_j + \alpha c_j) < -\tanh(ac_i) - \tanh(ac_j). \]

Making the substitutions \( x = ac_i, y = ac_j \) and \( z = ac \) we get
\[ \tanh(x - z) + \tanh(y + z) > \tanh(x) + \tanh(y), \]
with \( x > y > 0 \) and \( 0 < z < \frac{2\pi}{2}. \) Setting
\[ f(z) = (\tanh(x - z) - \tanh(x)) + (\tanh(y + z) - \tanh(y)), \]
we use the mean value theorem of differential calculus to prove that
\[ \tanh(x - z) - \tanh(x) = -zb \left( 1 - \tanh^2 \left( \frac{\theta_1}{2} \right) \right) \]
\[ \tanh(y + z) - \tanh(y) = zb \left( 1 - \tanh^2 \left( \frac{\theta_2}{2} \right) \right), \]
where \( x - z < \theta_1 < x \) and \( y < \theta_2 < y + z. \) However, because \( 1 - \tanh^2(x) \) is strictly decreasing for \( x > 0 \) and \( b > 0 \) because \( z < \frac{2\pi}{2} \iff y + z < x - z, \) it follows that
\[ f(z) = zb \left[ (1 - \tanh^2 \left( \frac{\theta_2}{2} \right)) - (1 - \tanh^2 \left( \frac{\theta_1}{2} \right)) \right] > 0. \]

7) \( -\tanh_{a,b} \) and D3:
Theorem A.7: \( -\tanh_{a,b} \) satisfies D3
\[ S(\alpha + \vec{c}) < S(\vec{c}), \alpha \in \mathbb{R}, \alpha > 0. \]
Proof: It is enough to show that \( \frac{\partial S(\alpha + \vec{c})}{\partial \alpha} < 0 \) as if the derivative of the measure with respect to the parameter \( \alpha \) is negative then any \( \alpha \) causes the measure to drop.
\[ \frac{\partial}{\partial \alpha} \left[ -\sum_j \tanh(ac + ac_j) \right] = -\sum_j \left( 1 - \tanh^2 ((ac + ac_j)) \right) b(ac + ac_j) \epsilon^{-1} a < 0, \]
which is true as \( a, b > 0 \) and \( \tanh^2 \theta < 1. \)

8) \( -\log \) and D3:
Proof: \( -\log \) satisfies
\[ S(\alpha + \vec{c}) < S(\vec{c}), \alpha \in \mathbb{R}, \alpha > 0. \]
\[ \leq \sum_j \log \left( \frac{1 + (ac + c_j)^2}{1 + c_j^2} \right) > 0, \]
Which is true because
\[ \frac{1 + (ac + c_j)^2}{1 + c_j^2} > 1, \alpha > 0. \]

9) \( k_4 \) and D2:
Theorem A.8: \( k_4 \) satisfies D2
\[ S(\alpha \vec{c}) = S(\vec{c}), \forall \alpha \in \mathbb{R}, \alpha > 0 \]
Proof:
\[ \frac{\sum_j (ac_j)^4}{\left( \sum_j (ac_j)^2 \right)^2} = \frac{\alpha^4 \sum_j c_j^4}{\left( \sum_j c_j^2 \right)^2} = \frac{\sum_j c_j^4}{\left( \sum_j c_j^2 \right)^2}. \]

10) \( k_4 \) and D3:
Theorem A.9: \( k_4 \) satisfies D3
\[ S(\alpha + \vec{c}) < S(\vec{c}), \alpha \in \mathbb{R}, \alpha > 0. \]
Proof: Set

\[ f(\alpha) = \frac{\sum_i (c_i + \alpha)^4}{\left(\sum_i (c_i + \alpha)^2\right)^2}. \]

It follows that

\[
\frac{\partial f}{\partial \alpha} = 4 \left[ \sum_i (c_i + \alpha)^3 \sum_i (c_i + \alpha)^2 - \sum_i (c_i + \alpha)^4 \sum_i (c_i + \alpha) \right] \left(\sum_i (c_i + \alpha)^2\right)^3.
\]

We can ignore the denominator as it is clearly positive. We claim that \( \frac{\partial f}{\partial \alpha} < 0 \) for \( \alpha > 0 \). This is because, for positive \( x_i \), it is always true that

\[
\sum_i x_i^2 \sum_i x_i^3 < \sum_i x_i^4 \sum_i x_i
\]

as

\[
\sum_i x_i^2 \sum_i x_i^3 - \sum_i x_i^4 \sum_i x_i = \sum_{i \neq j} x_i x_j [x_i x_j^2 + x_i^2 x_j - x_i^3 - x_j^3] = -\sum_{i \neq j} x_i x_j (x_i - x_j)^2 (x_i + x_j) < 0.
\]

11) \( \kappa_\beta \) and P1:

Theorem A.10: \( \kappa_\beta \) satisfies P1

\[ \forall i, \exists \beta_i > 0 \text{ such that } \forall \alpha > 0: \]

\[ S([c_1 \ldots c_i + \beta + \alpha \ldots]) > S([c_1 \ldots c_i + \beta \ldots]). \]

Proof: Fix \( i \) and make the substitution \( \tilde{c}_i = c_i + \beta \). We show that the derivative of the measure is positive and hence the measure increases for any \( \alpha \)

\[
\frac{\partial}{\partial \alpha} \left[ \sum_{i \neq i} c_j^4 + (\tilde{c}_i + \alpha)^4 \right] > 0.
\]

The numerator of the derivative is

\[
(\tilde{c}_i + \alpha)^3 \left( \sum_{j \neq i} c_j^2 + (\tilde{c}_i + \alpha)^2 \right)
\]

\[ - \left( \sum_{j \neq i} c_j^4 + (\tilde{c}_i + \alpha)^4 \right) (\tilde{c}_i + \alpha) > 0. \]

Multiplying out and substituting back in for \( \tilde{c}_i \) this becomes

\[ c_i + \alpha + \beta > \sqrt{\frac{\sum_{j \neq i} c_j^4}{\sum_{j \neq i} c_j^3}}. \]

Clearly there exists a \( \beta \) such that the above expression holds true for all \( \alpha > 0 \). \[ \square \]

12) \( -\theta_{\mathcal{N}} \) and P1:

Theorem A.11: \( -\theta_{\mathcal{N}} \) satisfies P1

\[ \forall i, \exists \beta_i > 0 \text{ such that } \forall \alpha > 0: \]

\[ S([c_1 \ldots c_i + \beta + \alpha \ldots]) > S([c_1 \ldots c_i + \beta \ldots]). \]

Proof: Without loss of generality we can change the conditions slightly by replacing \( p \) (\( p < 0 \)) with \( -p \) and correspondingly update the constraint to \( p > 0 \).

\[
-\sum_{j \neq i, c_j \neq 0} c_j^{-p} - (c_i + \beta + \alpha)^{-p} > -\sum_{j \neq i, c_j \neq 0} c_j^{-p} - (c_i + \beta)^{-p}
\]

\[
(c_i + \beta + \alpha)^{-p} < (c_i + \beta)^{-p}
\]

\[
\frac{1}{(c_i + \beta + \alpha)^p} < \frac{1}{(c_i + \beta)^p}
\]

which is true if \( \beta > 0 \). \[ \square \]

13) \( u_{\theta} \) and D1:

Theorem A.12: \( u_{\theta} \) does not satisfy D1

\[ S([c_1 \ldots c_i + \alpha \ldots c_j + \alpha \ldots]) > S(\tilde{c}), \]

for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2} \).

Proof: For \( \theta = .5 \)

\[ S([1, 2, 4, 9]) = .6667 \]

\[ S([1.1, 1.9, 4, 9]) = .7333, \]

The Robin Hood operation increased sparsity and hence does not satisfy D1. \[ \square \]

14) \( u_{\theta} \) and D3:

Theorem A.13: \( u_{\theta} \) does not satisfy D3

\[ S(\alpha + \tilde{c}) < S(\tilde{c}), \alpha \in \mathbb{R}, \alpha > 0. \]

Proof: The support of \( \tilde{c} \) is \([c_{(1)}, c_{(N)}]\). Assume the support of the \( \theta \mathcal{N} \) points that correspond to the minimum is \([c_{(k)}, c_{(j)}]\). By adding a constant, \( \alpha \), to each coefficient in the distribution we shift the distribution to \( \tilde{c} + \alpha \). Clearly, neither of the two supports mentioned above changes: \((c_{(j)} - \alpha) < (c_{(k)} - \alpha) = c_{(j)} - c_{(k)}\). Hence \( u_{\theta} \) does not satisfy D3. \[ \square \]

15) \( u_{\theta} \) and D4:

Theorem A.14: \( u_{\theta} \) satisfies D4

\[ S(\tilde{c}) = S(\tilde{c} | \tilde{c}) = S(\tilde{c} | \tilde{c}) = S(\tilde{c} | \tilde{c}) \cdots | \tilde{c}). \]

Proof: The support of \( \tilde{c} \) is \([c_{(1)}, c_{(N)}]\). Assume the support of the \( \theta \mathcal{N} \) points that correspond to the minimum is \([c_{(k)}, c_{(j)}]\). The new set, \([\tilde{c} | \tilde{c}]\), has \(2^{|\mathcal{N}}| \) points lying between \( c_{(j)} \) and \( c_{(k)} \), that is, neither of the previously mentioned two supports has changed. This reasoning holds for cloning the data more than once. Hence \( u_{\theta} \) satisfies D4. \[ \square \]

16) \( u_{\theta} \) and P1:

Theorem A.15: \( u_{\theta} \) satisfies P1
\[ \forall \alpha, \exists \beta = \beta_k > 0, \text{ such that } \forall \alpha > 0: \]

\[ S([c_1 \ldots c_i + \beta + \alpha \ldots]) > S([c_1 \ldots c_i + \beta \ldots]). \]

**Proof:** The support of \( \tilde{c} \) is \([c(1), c(N)]\). Without loss of generality we focus on \( c(N) \) as the effect of adding sufficiently large \( \beta \) to any other coefficient will result in this coefficient becoming the largest. We choose \( \beta \) sufficiently large so that \( c(N) + \beta \) is set sufficiently far apart from the other coefficients for the support of \( [\theta N] \) points that correspond to the minimum not to contain \( c(N) \). Consequently, the numerator of the minimization term is a constant \( K \) not depending on \( \beta \) or \( \alpha \). We can rewrite

\[ S([c_1 \ldots c_i + \beta + \alpha \ldots]) > S([c_1 \ldots c_i + \beta \ldots]) \]

as

\[ 1 - \frac{K}{c(N) - c(1) + \alpha + \beta} < 1 - \frac{K}{c(N) - c(1) + \alpha} \]

which is clearly true and the proof is complete. \( \square \)

17) \( u_\theta \) and P2:

**Theorem A.16:** \( u_\theta \) does not satisfy P2

\[ S(\tilde{c}|0) > S(\tilde{c}). \]

**Proof:** Assume \( \tilde{c} \) has total support \( c(N) = c(1) \) and the support of \( \[\theta N\] \) points lying between values \( c(j) \) and \( c(k) \). If \( 0 \) lies within the range \( c(j) < c(k) \), adding a \( 0 \) will decrease the range to \( c(j) \) without increasing the total support. \( \square \)

18) \( H_C \) and D1:

**Theorem A.17:** \( H_C \) satisfies D1

\[ S([c_1 \ldots c_i - \alpha \ldots c_j + \alpha \ldots]) < S(\tilde{c}), \]

for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2} \).

**Proof:**

\[ -\sum_{k \neq i,j} \ln c_k^2 - \ln (c_i - \alpha)^2 - \ln (c_j + \alpha)^2 < -\sum_k \ln c_k^2 \]

\[ -2\ln (c_i - \alpha) - 2\ln (c_j + \alpha) < -2\ln c_i - 2\ln c_j \]

\[ (c_i - \alpha)(c_j + \alpha) > c_i c_j \]

\[ \alpha < c_i - c_j, \]

which is clearly true. \( \square \)

19) Hoyer and D1:

**Theorem A.18:** Hoyer satisfies D1

\[ S([c_1 \ldots c_i - \alpha \ldots c_j + \alpha \ldots]) < S(\tilde{c}) \]

for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2} \).

**Proof:**

\[ \frac{\partial}{\partial \alpha} \left( \frac{\sqrt{N} - \sum_{i=1}^N (c_i + \alpha)}{\sqrt{N} - 1} \right) \]

\[ \equiv \frac{\partial}{\partial \alpha} \left[ \frac{-1}{\sqrt{N} - 1} \left( \sum_{k \neq i,j} (c_k^2 + (c_i - \alpha)^2 + (c_j + \alpha)^2)^{-\frac{1}{2}} \right) \right], \]

which is

\[ \sum_{k \neq i,j} c_j \]

\[ \frac{\sqrt{N} - 1}{\sqrt{N} - 1} \left( \sum_{i=1}^N \left( c_i + \alpha \right) \right) \left( \sum_{i=1}^N (c_i + \alpha)^2 \right)^{-\frac{1}{2}} \]

which is true as \( (c_j - c_i - 2\alpha) < 0 \) . \( \square \)

20) Hoyer and D3:

**Theorem A.19:** Hoyer satisfies D3

\[ S(\alpha + \tilde{c}) < S(\tilde{c}), \alpha \in \mathbb{R}, \alpha > 0. \]

**Proof:**

\[ \frac{\partial}{\partial \alpha} \left( \frac{\sqrt{N} - \sum_{i=1}^N (c_i + \alpha)}{\sqrt{N} - 1} \right) \]

\[ \equiv \frac{\partial}{\partial \alpha} \left[ \frac{-1}{\sqrt{N} - 1} \left( \sum_{i=1}^N (c_i + \alpha) \right) \left( \sum_{i=1}^N (c_i + \alpha)^2 \right)^{-\frac{1}{2}} \right], \]

With the substitution

\[ s_1 = \sum_{i=1}^N c_i \]

\[ s_2 = \sum_{i=1}^N c_i^2 \]

this becomes

\[ (s_1 + N\alpha)^2(s_2 + 2s_1 + N\alpha^2)^{-\frac{1}{2}} - N(s_2 + 2s_1 + N\alpha^2) < 0 \]

which simplifies to

\[ N > \frac{s_1^2}{s_2}. \]

We rewrite this as

\[ Ns_2 = \sum_{i=1}^N \sum_{i=1}^N c_i^2 > \left( \sum_{i=1}^N c_i \right)^2 = s_1 \]

which is true by Cauchy–Schwarz. \( \square \)
21) Hoyer and P1:

Theorem A.20: Hoyer satisfies P1

∀ε, ∃β = β_k > 0, such that ∀α > 0:

\[ S([c_1 \ldots c_i + \beta + \alpha \ldots]) > S([c_1 \ldots c_i + \beta \ldots]). \]

Proof:

\[
\frac{\partial}{\partial \alpha} \left( \sqrt{N} - \frac{1}{2} \right) = \alpha \left[ \left( \sum_{i \neq j} c_i \right) (c_j + \alpha + \beta) - \sum_{j \neq i} c_j^2 \right].
\]

which is

\[
\left( \frac{\sum_{j \neq i} c_j^2 + c_j + \alpha + \beta}{\sqrt{N} - 1} \right)^{\frac{3}{2}} \times \left( \sum_{j \neq i} c_j \right) (c_j + \alpha + \beta - \sum_{j \neq i} c_j^2).
\]

Clearly for sufficiently large β the above quantity is > 0. □

22) pq-mean and D2:

Theorem A.21: The pq-mean satisfies D2

\[ S(\alpha c) = S(c), \forall \alpha \in \mathbb{R}, \alpha > 0. \]

Proof:

\[
\left( \frac{\frac{1}{n} \left( \sum_i (\alpha c_i) p^\frac{1}{p} \right)^{\frac{1}{p}}}{\frac{1}{n} \sum_i \alpha c_i q^\frac{1}{q}} \right)^{\frac{1}{p}} = \left( \frac{\frac{c^p}{n} \left( \sum_i c_i^p \right)^{\frac{1}{p}}}{\frac{c}{n} \sum_i c_i^q} \right)^{\frac{1}{p}} \cdot \left( \frac{\frac{1}{n} \sum_i \alpha c_i q^\frac{1}{q}}{\frac{1}{n} \sum_i c_i q^\frac{1}{q}} \right)^{\frac{1}{p}}.
\]

23) pq-mean and D3:

Theorem A.22: The pq-mean satisfies D3

\[ S(\alpha + \tilde{c}) < S(\tilde{c}), \alpha \in \mathbb{R}, \alpha > 0. \]

Proof: We make the following substitutions:

\[ \|\tilde{c}\|_p = \left( \sum_i c_i^p \right)^{\frac{1}{p}}, \]

\[ \|\tilde{c} + \alpha\|_p = \left( \sum_i (c_i + \alpha)^p \right)^{\frac{1}{p}}. \]

We want to show

\[
\left( \frac{1}{N} \frac{\|\tilde{c} + \alpha\|_p}{\|\tilde{c}\|_q} \right)^{\frac{1}{p}} < \left( \frac{1}{N} \frac{\|\tilde{c}\|_p}{\|\tilde{c}\|_q} \right)^{\frac{1}{p}} \cdot \left( \frac{\sum_i c_i^p}{\sum_i c_i^q} \right)^{\frac{1}{p}} \cdot \left( \frac{\sum_i c_i^q}{\sum_i c_i^q} \right)^{\frac{1}{p}}.
\]

which we can rearrange to get

\[ \|\tilde{c} + \alpha\|_p \|\tilde{c}\|_q - \|\tilde{c} + \alpha\|_p \|\tilde{c}\|_q > 0. \]

Using the identity

\[ (1 + \epsilon)^p \approx 1 + p\epsilon \]

for ε small we can write

\[
\left( \sum_i (c_i + \alpha)^p \right)^{\frac{1}{p}} \approx \left( \sum_i c_i^p \left( 1 + \frac{\alpha}{c_i} \right)^p \right)^{\frac{1}{p}} \approx \left( \sum_i c_i^p \left( 1 + \frac{p\alpha}{c_i} \right)^{\frac{1}{p}} \right) \approx \left( \sum_i c_i^p \left( 1 + p\alpha \sum_i c_i^p \right) \right)^{\frac{1}{p}} \approx \|\tilde{c}\|_p \left( 1 + \alpha \sum_i c_i^p \right)^{\frac{1}{p}} \cdot (9)
\]

Substituting this into the inequality above (and letting \( \hat{\alpha} = \alpha \|\tilde{c}\|_p \|\tilde{c}\|_q \)):

\[
\|\tilde{c} + \alpha\|_p \|\tilde{c}\|_q - \|\tilde{c} + \alpha\|_p \|\tilde{c}\|_q \approx \|\tilde{c}\|_p \|\tilde{c}\|_q \left( 1 + \alpha \sum_i c_i^p \right) \cdot (7)
\]

\[
\|\tilde{c}\|_p \|\tilde{c}\|_q \left( 1 + \alpha \sum_i c_i^p \right) - \|\tilde{c}\|_p \|\tilde{c}\|_q \left( 1 + \alpha \sum_i c_i^p \right) = \hat{\alpha} \left( \sum_i c_i^p \left( 1 - \sum_i c_i^p \right) - \sum_i c_i^p \right) \cdot (9)
\]

which is positive as

\[
\begin{align*}
&c_j - c_i > 0 \iff c_i^{p-1} c_j^{p-1} - c_i c_j^{p-1} > 0 \\
c_j - c_i < 0 \iff c_i c_j^{p-1} - c_i^{p-1} c_j^{p-1} < 0.
\end{align*}
\]

The approximations above guarantee the result to be true for small α, namely the existence of an \( \alpha_{\text{max}} \) such that the result holds true for all \( \alpha \in [0, \alpha_{\text{max}}] \). To extend the validity of the result to all \( \alpha \), we first determine \( n \) such that \( \frac{n}{p} \) lies in the range above. We then repeat the argument \( n \) times, in step \( k \) adding \( \frac{n}{k} \) to vector \( \tilde{c} + (k - 1)\frac{n}{k} \). We argue that \( \alpha_{\text{max}} \) does not decrease from one step to the next. This is because it is determined by the ratios \( \frac{n}{k} \), and the \( c_j \) increase from one step to the next. □
24) pq-mean and D4:

Theorem A.23: The pq-mean satisfies D4

\[ S(\bar{c}) = S(\|\bar{c}\|) = S(\|\bar{c}\|, \|\bar{c}\|) = S(\|\bar{c}\|, \|\bar{c}\|, \ldots, \|\bar{c}\|). \]

Proof: We clone \( \bar{c} \) \( M \) times to get the vector \( \bar{d} \) which has length \( MN \)

\[ S(M\|\bar{c}\|) = S(\bar{d}) \]

\[ = -\left( \frac{1}{MN} \sum_{i=1}^{MN} c_i \right)^\frac{1}{p} \]

\[ = -\left( \frac{M}{MN} \sum_{i=1}^{N} c_i^p \right)^\frac{1}{p} \]

\[ = -\left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} c_i^p \right)^\frac{1}{p} \]

We want to show

\[ -\left( \frac{1}{N} \right)^{\frac{1}{p}} \|c(\bar{c}, \alpha)\|_p < -\left( \frac{1}{N} \right)^{\frac{1}{p}} \|c(\bar{c}, \alpha)\|_q \]

which we rewrite

\[ \|c(\bar{c}, \alpha)\|_p < \|c(\bar{c}, \alpha)\|_q > 0, \]

Substituting the terms from (10), along with the equivalent \( q \) term gives

\[ \|c(\bar{c}, \alpha)\|_q ( \|c(\bar{c}, \alpha)\|_p + \frac{\alpha(c_1 + \beta)^{p-1}}{\|c(\bar{c}, \alpha)\|_p} ) -\|c(\bar{c}, \alpha)\|_p \left( \|c(\bar{c}, \alpha)\|_q + \frac{\alpha(c_1 + \beta)^{p-1}}{\|c(\bar{c}, \alpha)\|_q} \right) > 0, \]

which simplifies to

\[ (c_1 + \beta)^{q-p} > \|c(\bar{c}, \alpha)\|_p \]

If we examine the RHS further

\[ \|c(\bar{c}, \alpha)\|_q ( \|c(\bar{c}, \alpha)\|_p + \frac{\alpha(c_1 + \beta)^{p-1}}{\|c(\bar{c}, \alpha)\|_p} ) \]

\[ = (c_1 + \beta)^{q-p} + \frac{\alpha(c_1 + \beta)^{p-1}}{c_1 + \beta}. \]

To prove the theorem we need

\[ \frac{\sum_{i>1} c_i}{(c_1 + \beta)^p} < \frac{\sum_{i>1} c_i}{(c_1 + \beta)^p} \]

which is equivalent to

\[ (c_1 + \beta)^{q-p} > \sum_{i>1} c_i \]

which is true for a sufficiently large \( \beta \).

The result for arbitrary \( \alpha \) is obtained by repeating the argument in Theorem A.22. \( \square \)

25) pq-mean and P1:

Theorem A.24: The pq-mean satisfies P1

\( \forall \alpha, \exists \beta = \beta_0 > 0 \), such that \( \forall \alpha > 0 \):

\[ S([c_1 \ldots c_i + \beta + \alpha \ldots]) > S([c_1 \ldots c_i + \alpha \ldots]). \]

Proof: Without loss of generality we operate on the first coefficient \( c_1 \) and define

\[ c(\beta, \alpha) = [c_1 + \beta + \alpha \quad c_2 \quad \ldots \quad c_N] \]

and

\[ \|\bar{c}\|_p = \left( \sum_{i} c_i^p \right)^\frac{1}{p}. \]

For a small \( \alpha \) we can write

\[ (c_1 + \beta + \alpha)^p = (c_1 + \beta)^p \left( 1 + \frac{\alpha}{c_1 + \beta} \right)^p \]

\[ \approx (c_1 + \beta)^p \left( 1 + \frac{p\alpha}{c_1 + \beta} \right) \]

\[ = (c_1 + \beta)^p + p\alpha(c_1 + \beta)^{p-1}. \]

Using this expression we get

\[ \|c(\beta, \alpha)\|_p = \left( (c_1 + \beta + \alpha)^p + \sum_{i>1} c_i^p \right)^\frac{1}{p} \]

\[ \approx \left( (c_1 + \beta)^p + p\alpha(c_1 + \beta)^{p-1} + \sum_{i>1} c_i^p \right)^\frac{1}{p} \]

\[ = \|c(\beta, 0)\|_p \left( 1 + \frac{\alpha(c_1 + \beta)^{p-1}}{\|c(\beta, 0)\|_p} \right) \]

\[ = \|c(\beta, 0)\|_p + \frac{\alpha(c_1 + \beta)^{p-1}}{\|c(\beta, 0)\|_p}. \]

26) pq-mean and P2:

Theorem A.25: The pq-mean satisfies P2

\[ S(\bar{c})(0) > S(\bar{c}). \]

Proof: We make the following substitution

\[ \|\bar{c}\|_p = \left( \sum_{i=1}^{N} c_i^p \right)^\frac{1}{p}. \]
We need
\[
- \frac{\left( \frac{1}{N+1} \sum_{i=1}^{N+1} c_i \right)^{\frac{1}{\beta}}}{\left( \frac{1}{N+1} \sum_{i=1}^{N+1} c_i \right)^{\frac{1}{\beta}}}
= - \frac{\left( \frac{1}{N+1} \sum_{i=1}^{N} c_i^p + 0p \right)^{\frac{1}{\beta}}}{\left( \frac{1}{N+1} \sum_{i=1}^{N} c_i^q + 0q \right)^{\frac{1}{\beta}}}
= - \frac{\left( \frac{1}{N+1} \sum_{i=1}^{N} \frac{1}{\beta} |c_i|^p \right)^{\frac{1}{\beta}}}{\left( \frac{1}{N+1} \sum_{i=1}^{N} \frac{1}{\beta} |c_i|^q \right)^{\frac{1}{\beta}}}
= - \frac{\left( \frac{1}{N+1} \sum_{i=1}^{N+1} |c_i| \right)^{\frac{1}{\beta}}}{\left( \frac{1}{N+1} \sum_{i=1}^{N+1} |c_i| \right)^{\frac{1}{\beta}}}
>rac{\left( \frac{1}{N+1} \sum_{i=1}^{N} |c_i| \right)^{\frac{1}{\beta}}}{\left( \frac{1}{N+1} \sum_{i=1}^{N} |c_i| \right)^{\frac{1}{\beta}}}
which is true if
\[
\left( \frac{1}{N+1} \sum_{i=1}^{N} \frac{1}{\beta} |c_i| \right)^{\frac{1}{\beta}} = \left( \frac{N}{N+1} \right)^{\frac{1}{\beta}-\frac{1}{\beta}} < 1.
\]
This is true for \( p < q \).

27) \( pq \)-mean and D1:

**Theorem A.26**: The \( pq \)-mean satisfies D1

\[ S(c_1, \ldots, c_i - \alpha, \ldots, c_j + \alpha, \ldots) < S(c) \]
for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2} \) if \( p \leq 1, q > 1 \).

**Proof**: We define
\[
f(\bar{c}) = - \left( \sum_{k=1}^{N} c_k^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{N} c_k^q \right)^{\frac{1}{q}}
\]
and determine the partial derivatives
\[
\frac{\partial f}{\partial c_k} = - a(\bar{c}) c_k^{p-1} + b(\bar{c}) c_k^{q-1}
\]
\[
\frac{\partial f}{\partial c_j} = - a(\bar{c}) c_j^{p-1} + b(\bar{c}) c_j^{q-1}
\]
where
\[
a(\bar{c}) = \left( \sum_{k=1}^{N} c_k^p \right)^{-\frac{1}{p}} \left( \sum_{k=1}^{N} c_k^q \right)^{-\frac{1}{q}}
\]
\[
b(\bar{c}) = \left( \sum_{k=1}^{N} c_k^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{N} c_k^q \right)^{\frac{1}{q}}
\]
When \( p \leq 1, q > 1 \) and \( c_i > c_j \),
\[
c_i^{p-1} > c_j^{q-1}
\]
and
\[
c_i^{p-1} > c_j^{q-1}
\]
therefore
\[
\frac{\partial f}{\partial c_i} > \frac{\partial f}{\partial c_j},
\]
that is, a decrease of \( c_i \) by a small amount with a simultaneous increase of \( c_j \) by the same amount will lead to the decrease of the function \( f \).

28) Gini and D1:

**Theorem A.27**: The Gini Index satisfies D1

\[ S(c_1, \ldots, c_i - \alpha, \ldots, c_j + \alpha, \ldots) < S(c) \]
for all \( \alpha, c_i, c_j \) such that \( c_i > c_j \) and \( 0 < \alpha < \frac{c_i - c_j}{2} \).

**Proof**: The Gini Index of \( \bar{c} = [c_1, c_2, c_3, \ldots] \) is given by
\[
S(\bar{c}) = 1 - 2 \sum_{k=1}^{N} \frac{c(k)}{|\bar{c}|_1} \left( \frac{N-k+\frac{1}{2}}{N} \right),
\]
where \( (k) \) denotes the new index after sorting from lowest to highest, that is, \( c_1 \leq c_2 \leq \cdots \leq c_N \).

Without loss of generality we can assume that the two coefficients involved in the Robin Hood operation are \( c(i) \) and \( c(j) \). After a Robin Hood operation is performed on \( \bar{c} \), we label the resulting set of coefficients \( \bar{d} \) which are sorted using an index which we denote \([1, \ldots, N] \), that is, \( d[1] \leq d[2] \leq \cdots \leq d[N] \). Let us assume that the Robin Hood operation alters the sorted ordering in that the new coefficient obtained by the subtraction of \( \alpha \) from \( c(i) \) has the new rank \( i-n \), that is,
\[
d[i-n] = c(i) - \alpha
\]
and the new coefficient obtained by the addition of \( \alpha \) to \( c(j) \) has the new rank \( j+m \), that is,
\[
d[j+m] = c(j) + \alpha.
\]
The correspondence between the coefficients of \( \bar{c} \) and \( \bar{d} \) is shown in Fig. 4 and in mathematical terms is
\[
d[k] = c(k) \quad \text{for} \quad 1 \leq k \leq j - 1
\]
\[
d[k] = c(k+1) \quad \text{for} \quad j \leq k \leq j + m - 1
\]
\[
d[k] = c(j) + \alpha \quad \text{for} \quad k = j + m
\]
\[
d[k] = c(k) \quad \text{for} \quad j + m + 1 \leq k \leq i - n - 1
\]
\[
d[k] = c(i) - \alpha \quad \text{for} \quad k = i - n
\]
\[
d[k] = c(k-1) \quad \text{for} \quad i - n + 1 \leq k \leq i
\]
\[
d[k] = c(k) \quad \text{for} \quad i + 1 \leq k \leq N.
\]
We wish to show
\[
S(\bar{d}) > S(\bar{c}).
\]
Removing common terms and noting that $\|\vec{c}\|_1 = \|\vec{d}\|_1$ we can simplify this to

$$\sum_{k \in \Delta} c_{(k)} \left( N - k + \frac{1}{2} \right) < \sum_{k \in \Delta} d_{[k]} \left( N - k + \frac{1}{2} \right),$$

where $\Delta = \{j, j+1, \ldots, j+m, i-n, i-n+1, \ldots \}$. Using the correspondence above we can express the coefficients of $\vec{d}$ in terms of the coefficients of $\vec{c}$. We then get

$$\sum_{k=1}^{m} c_{(j+k)} \left[ \left( N - j - k + 1 + \frac{1}{2} \right) - \left( N - j - k + \frac{1}{2} \right) \right] + \sum_{k=1}^{n} c_{(i-k)} \left[ \left( N - i + k - 1 + \frac{1}{2} \right) - \left( N - i + k + \frac{1}{2} \right) \right] + c_{(j)} \left[ \left( N - j - m + 1 + \frac{1}{2} \right) - \left( N - j + 1 \right) \right] + c_{(j)} \left[ \left( N - i + n + 1 + \frac{1}{2} \right) - \left( N - i + n + \frac{1}{2} \right) \right] + \alpha \left[ \left( N - j - m + 1 + \frac{1}{2} \right) - \left( N - i + n + \frac{1}{2} \right) \right] > 0$$

which becomes

$$\sum_{k=1}^{m} \left( c_{(j+k)} - c_{(j)} \right) + \sum_{k=1}^{n} \left( c_{(i-k)} - c_{(i-k)} \right) + \alpha \left( (i-n) - (j+m) \right) > 0.$$ 

This is true as the two summations are positive as the negative component has a lower sorted index than the positive and is hence smaller and the last term is positive due to the condition on $\alpha$.

**Proof:** Rewriting $S(\vec{\alpha} + \vec{c}) < S(\vec{c})$ and making the substitution

$$f(k) = \left( \frac{N - k + \frac{1}{2}}{N} \right)$$

we get the following:

$$\sum_{k=1}^{N} \frac{c_{(k)}}{\|\vec{c} + \alpha\|_1} f(k) + \frac{N \alpha}{\|\vec{c} + \alpha\|_1} \sum_{k=1}^{N} f(k) - \sum_{k=1}^{N} \frac{c_{(k)}}{\|\vec{d}\|_1} f(k) > 0$$

$$\sum_{k=1}^{N} \frac{c_{(k)}}{\|\vec{c} + \alpha\|_1} f(k) \left( \frac{1}{\|\vec{c} + \alpha\|_1} - \frac{1}{\|\vec{d}\|_1} \right) + \frac{N \alpha}{\|\vec{c} + \alpha\|_1} \sum_{k=1}^{N} f(k) > 0$$

$$\sum_{k=1}^{N} \frac{c_{(k)}}{\|\vec{c} + \alpha\|_1} f(k) \left( \frac{-N \alpha}{\|\vec{c} + \alpha\|_1} \right) + \frac{N \alpha}{\|\vec{c} + \alpha\|_1} \sum_{k=1}^{N} f(k) > 0$$

$$\sum_{k=1}^{N} f(k) \left( 1 - \frac{c_{(k)}}{\|\vec{d}\|_1} \right) > 0.$$ 

This is clearly true for $N > 1$.

**31) Gini and D4:**

**Theorem A.30:** The Gini Index satisfies D4

$$S(\vec{c}) = S(\vec{c}||\vec{c}) = S(\vec{c}||\vec{c}) = S(\vec{c}||\vec{c}||\vec{c}) = \cdots = S(\vec{c}).$$

**Proof:** We clone $\vec{c}$ $M$ times to get the vector $\vec{d}$ which has length $MN$

$$S(\vec{c}||\vec{c}||\vec{c}) \stackrel{M}{\longleftarrow} \vec{d}$$

$$S(\vec{d}) = S(\vec{d})$$

$$= 1 - 2 \sum_{k=1}^{MN} \frac{d_{[k]}}{\|\vec{d}\|_1} \left( \frac{MN - k + \frac{1}{2}}{MN} \right)$$

$$= 1 - 2 \sum_{j=1}^{M} \sum_{i=1}^{N} \frac{c_{(k)}}{\|\vec{c}\|_1} \left( \frac{MN - (Mi - M + j + 1)}{MN} \right)$$

$$= 1 - 2 \sum_{j=1}^{M} \frac{c_{(k)}}{\|\vec{c}\|_1} \left( \frac{MN - M + j + 1}{M^2N} \right)$$

$$= 1 - 2 \sum_{j=1}^{M} \frac{c_{(k)}}{\|\vec{c}\|_1} \left( \frac{M^2N - M^2i + M^2 - (M^2i + M + \frac{1}{2})}{M^2N} \right)$$

$$= 1 - 2 \sum_{j=1}^{M} \frac{c_{(k)}}{\|\vec{c}\|_1} \left( \frac{M^2N - M^2i + M^2 - \frac{M^2}{2} - \frac{M^2}{2} + \frac{M^2}{2}}{M^2N} \right)$$

$$= 1 - 2 \sum_{j=1}^{M} \frac{c_{(k)}}{\|\vec{c}\|_1} \left( \frac{M^2N - M^2i + M^2}{M^2N} \right)$$

$$= 1 - 2 \sum_{j=1}^{M} \frac{c_{(k)}}{\|\vec{c}\|_1} \left( \frac{N - i + j + \frac{1}{2}}{N} \right)$$

$$= S(\vec{c}).$$
32) Gini and P1:

Theorem A.31: The Gini Index satisfies P1
\( \forall \alpha, \exists \beta = \beta > 0, \) such that \( \forall \alpha > 0: \)

\[ S(\{c_1, \ldots, c_i + \beta + \alpha \ldots\}) > S(\{c_1, \ldots, c_i + \beta \ldots\}). \]

Proof: We use the following notation,

\[ \tilde{c} = \{c(1), c(2), \ldots, c(N) + \beta\}. \]

Without loss of generality we have chosen to perform the operation on \( c(N) \) as \( \beta \) can absorb the additive value needed to change any of the \( c(i) \) to \( c(N) \).

We wish to show that

\[ 1 - 2 \sum_{i=1}^{N} \frac{c(i)}{||\tilde{c}||_1} \left( \frac{N - i + \frac{1}{2}}{N} \right) \]

\[ < 1 - 2 \sum_{i=1}^{N} \frac{c(i)}{||\tilde{c}||_1 + \beta} \left( \frac{N - i + \frac{1}{2}}{N} \right) - \frac{\beta}{N(||\tilde{c}||_1 + \beta)}. \]

We can simplify the above to

\[ \sum_{i=1}^{N} c(i) \left( \frac{N - i + \frac{1}{2}}{N} \right) \left( \frac{1}{||\tilde{c}||_1} - \frac{1}{||\tilde{c}||_1 + \beta} \right) > \frac{\beta}{2} \sum_{i=1}^{N} c(i) \left( N - i + \frac{1}{2} \right) \]

\[ \sum_{i=1}^{N} c(i) \left( N - i + \frac{1}{2} \right) > \frac{1}{2} \sum_{i=1}^{N} c(i) \left( N - i \right) > 0. \]

Hence, the Gini Index satisfies P1. \( \square \)

33) Gini and P2:

Theorem A.32: The Gini Index satisfies P2

\[ S(\tilde{c}) = S(\{c, 0\}) > S(\tilde{c}). \]

Proof: Let us define

\[ \tilde{d} = \tilde{c} | 0 = [c_1, c_2, c_3, \ldots, c_N, 0] \]

and we note that \( ||\tilde{d}||_1 = ||\tilde{c}||_1 \). Without loss of generality we assign the lowest rank to the added coefficient 0, that is, \( d_{N+1} = d_{(1)} \). We can now make the assertion \( d_{i+1} = c(i) \), yielding

\[ S(\tilde{d}) = 1 - 2 \sum_{k=2}^{N+1} \frac{d(k)}{||\tilde{d}||} \left( \frac{N + 1 - k + \frac{1}{2}}{N + 1} \right) \]

\[ - \frac{2}{||\tilde{d}||} \left( \frac{N + 1 - k + \frac{1}{2}}{N + 1} \right) . \]

Making the substitution \( i = k - 1 \) we get

\[ S(\tilde{d}) = 1 - 2 \sum_{i=1}^{N} \frac{d(i+1)}{||\tilde{d}||} \left( \frac{N + 1 - i + \frac{1}{2}}{N + 1} \right) \]

\[ = 1 - 2 \sum_{i=1}^{N} \frac{c(i)}{||\tilde{c}||} \left( \frac{N - i + \frac{1}{2}}{N + 1} \right) \]

\[ > 1 - 2 \sum_{i=1}^{N} \frac{c(i)}{||\tilde{c}||} \left( \frac{N - i + \frac{1}{2}}{N} \right) \]

\[ = S(\tilde{c}). \]

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References

HURLEY AND RICKARD: COMPARING MEASURES OF SPARSITY


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