

Transformations en ondelettes 2D directionnelles

— Un panorama

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Local applications

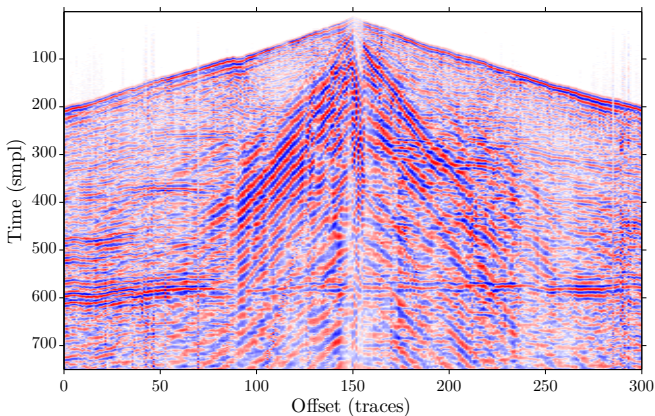


Figure: Geophysics: surface wave removal (before)

Local applications

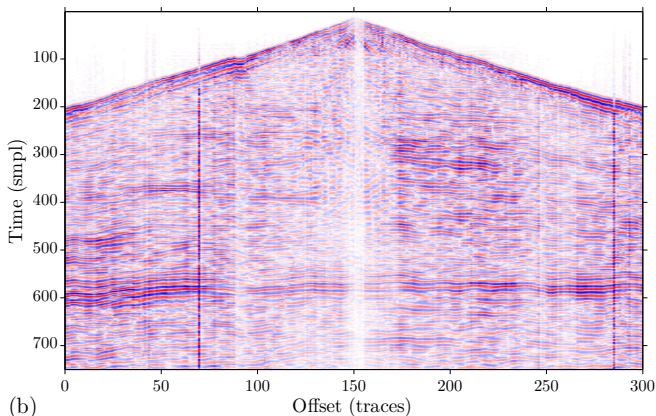


Figure: Geophysics: surface wave removal (after)

Agenda

- ▶ A quindeccennial panorama of improvements (≥ 1998)
 - ▶ sparser representations of contours and textures through increased spatial, directional and frequency selectivity
 - ▶ from fixed to adaptive, from low to high redundancy
 - ▶ generally fast, compact, informative, practical
 - ▶ lots of hybridization and small paths
- ▶ Outline
 - ▶ introduction
 - ▶ early days
 - ▶ oriented & geometrical:
 - ▶ directional, \pm separable (Hilbert/dual-tree)
 - ▶ directional, non-separable (conic)
 - ▶ directional, anisotropic scaling (curvelet, contourlet)
 - ▶ adaptive: lifting
 - ▶ conclusions

In just one slide

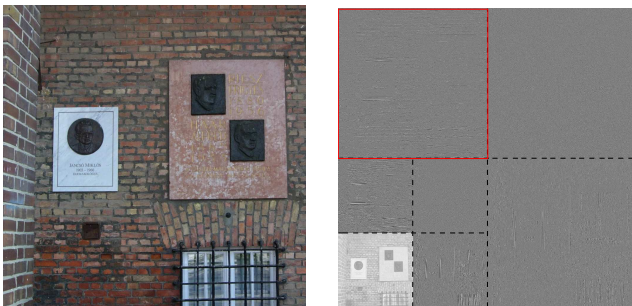


Figure: A standard, “dyadic”, separable wavelet decomposition

Where do we go from here? 15 years, 300+ refs in 30 minutes

Images are pixels (but...):

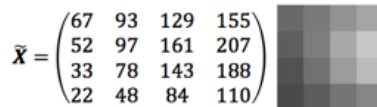


Figure: Image as a linear combination of pixels

- ▶ suffices for (simple) data (simple) manipulation
 - ▶ counting, enhancement, filtering
- ▶ very limited in higher level understanding tasks

Images are pixels (but...):

A review in an active research field:

- ▶ (partly) inspired by:
 - ▶ early vision observations
 - ▶ sparse coding: wavelet-like oriented filters and receptive fields of simple cells (visual cortex), [Olshausen *et al.*]
 - ▶ a widespread belief in sparsity
- ▶ motivated by image handling (esp. compression)
- ▶ continued from the first successes of wavelets
- ▶ aimed either at pragmatic or heuristic purposes
 - ▶ known formation model or unknown information
- ▶ developed through a quantity of *-lets and relatives

Images are pixels, wavelets are legion

Room(let) for improvement:

*Activelet, AMlet, Armlet, Bandlet, Barlet, Bathlet, Beamlet, Binlet, Bumplet, Brushlet, Caplet, Camplet, Chirplet, Chordlet, Circlet, Coiflet, Contourlet, Cooklet, Craplet, Cubelet, CURElet, Curvelet, Daublet, Directionlet, Dreamlet, Edgelet, FAMlet, FLaglet, Flatlet, Fourierlet, Framelet, Fresnelet, Gaborlet, GAMlet, Gausslet, Graphlet, Grouplet, Haarlet, Haardlet, Heatlet, Hutlet, Hyperbolet, Icalet (Icalette), Interpolet, Loglet, Marrlet, MIMOlet, Monowavelet, Morelet, **Morphlet**, Multiselectivelet, Multiwavelet, Needlet, Noiselet, Ondelette, Ondulette, Prewavelet, Phaselet, Planelet, Platelet, Purelet, QVlet, Radonlet, RAMlet, Randlet, Ranklet, Ridgelet, Riezlet, Ripplet (original, type-I and II), Scalet, S2let, Seamlet, Seislet, Shadelet, Shapelet, Shearlet, Sincllet, Singlet, Slantlet, Smoothlet, Snakelet, SOHOlet, Sparselet, Spikelet, Splinelet, Starlet, Steerlet, Stockeslet, SURE-let (SURElet), Surfacelet, Surflet, Symmlet, S2let, Tetrolet, Treelet, Vaguelette, Wavelet-Vaguelette, Wavelet, Warplet, Warplet, Wedgelet, Xlet, **who's next?***

Now, some reasons behind the quantity



Figure: Different kinds of images

Images are pixels, but different

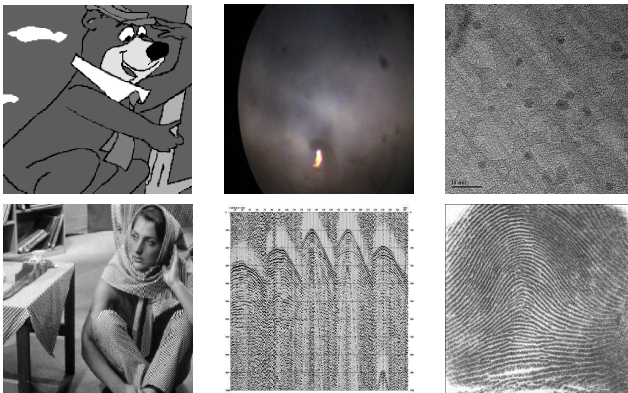


Figure: Different kinds of images

- 4 0001

1. *Journal of Management Studies*, 1990, 27, 1.

- DOI: 10.1002/anie.200511005

-

Images are pixels, but resolution/scale helps

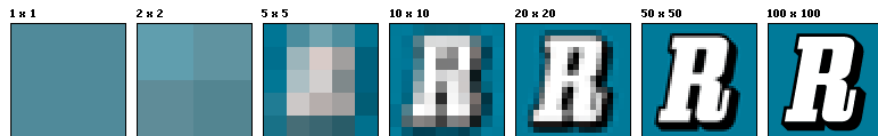


Figure: RRRrrrr: coarse to fine! [Chabat *et al.*, 2004]

- ▶ notion of sufficient resolution for understanding
- ▶ coarse-to-fine and fine-to-coarse links
- ▶ impact on more complex images?

Images are pixels, but deceiving



Figure: Real world image and illusions

Images are pixels, but resolution/scale helps

Important influence of the context

- ▶ use of scales/multiresolution associated with...
- ▶ a variety of methods for description/detection/modeling
 - ▶ smooth curve or polynomial fit, oriented regularized derivatives, discrete (lines) geometry, parametric curve detectors (such as the Hough transform), mathematical morphology, empirical mode decomposition, local *frequency estimators*, Hilbert and Riesz, Clifford algebras, optical flow approaches, smoothed random models, generalized Gaussian mixtures, hierarchical bounds, etc.



Figure: Memorial plaque in honor of A. Haar and F. Riesz: *A szegedi matematikai iskola világhírű megalapítói*, court. Prof. K. Szatmáry

Guiding thread (GT): early days

Fourier approach: critical, orthogonal

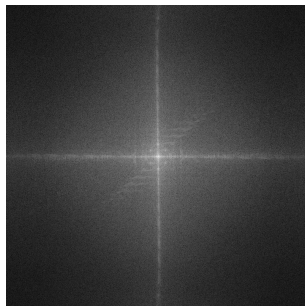


Figure: GT luminance component amplitude spectrum (log-scale)

Fast, compact, practical but not quite informative (no scale)

Scale-space approach: (highly)-redundant, more local

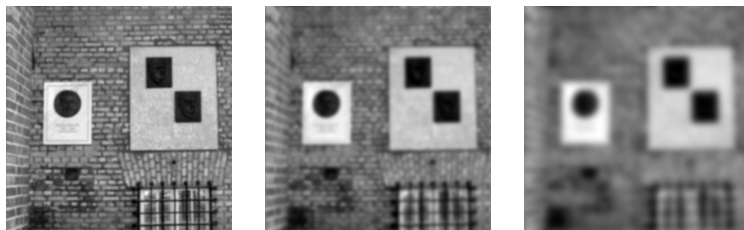


Figure: GT with Gaussian scale-space decomposition

Gaussian filters and heat diffusion interpretation
Varying persistence of features across scales

Scale-space approach: (less)-redundant, more local

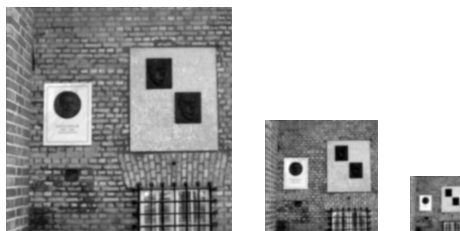


Figure: GT with Gaussian scale-space decomposition

Gaussian filters and heat diffusion interpretation

Varying persistence of features across scales + subsampling

Guiding thread (GT): early days

Differences in scale-space with subsampling

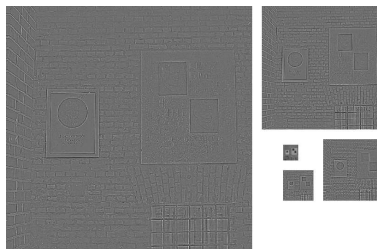


Figure: GT with Laplacian pyramid decomposition

Laplacian pyramid: complete, reduced redundancy, enhances image singularities, low activity regions/small coefficients, **algorithmic**

Guiding thread (GT): early days

Isotropic wavelets (more **axiomatic**)

Consider

Wavelet $\psi \in \mathbb{L}^2(\mathbb{R}^2)$ such that $\psi(\mathbf{x}) = \psi_{\text{rad}}(\|\mathbf{x}\|)$, with $\mathbf{x} = (x_1, x_2)$, for some radial function $\psi_{\text{rad}} : \mathbb{R}_+ \rightarrow \mathbb{R}$ (with adm. conditions).

Decomposition and reconstruction

For $\psi_{(\mathbf{b},a)}(\mathbf{x}) = \frac{1}{a}\psi(\frac{\mathbf{x}-\mathbf{b}}{a})$, $W_f(\mathbf{b}, a) = \langle \psi_{(\mathbf{b},a)}, f \rangle$ with reconstruction:

$$f(\mathbf{x}) = \frac{2\pi}{c_\psi} \int_0^{+\infty} \int_{\mathbb{R}^2} W_f(\mathbf{b}, a) \psi_{(\mathbf{b},a)}(\mathbf{x}) d^2\mathbf{b} \frac{da}{a^3} \quad (1)$$

if $c_\psi = (2\pi)^2 \int_{\mathbb{R}^2} |\hat{\psi}(\mathbf{k})|^2 / \|\mathbf{k}\|^2 d^2\mathbf{k} < \infty$.

Guiding thread (GT): early days

Multiscale edge detector, more potential wavelet shapes (DoG, Cauchy, etc.)

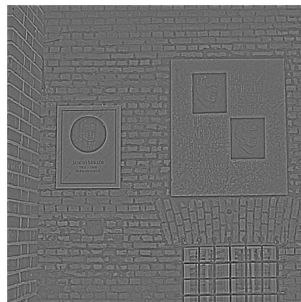
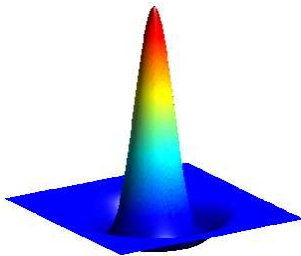


Figure: Example: Marr wavelet as a singularity detector

Guiding thread (GT): early days

Definition

The family \mathcal{B} is a frame if there exist two constants $0 < \mu_1 \leq \mu_2 < \infty$ such that for all f

$$\mu_1 \|f\|^2 \leq \sum_m |\langle \psi_m, f \rangle|^2 \leq \mu_2 \|f\|^2$$

Possibility of discrete orthogonal bases with $O(N)$ speed. In 2D:

Definition

Separable orthogonal wavelets: dyadic scalings and translations $\psi_m(\mathbf{x}) = 2^{-j} \psi^k(2^{-j} \mathbf{x} - \mathbf{n})$ of three tensor-product 2-D wavelets

$$\psi^V(\mathbf{x}) = \psi(x_1)\varphi(x_2), \psi^H(\mathbf{x}) = \varphi(x_1)\psi(x_2), \psi^D(\mathbf{x}) = \psi(x_1)\psi(x_2)$$

Oriented, \pm separable

To tackle orthogonal DWT limitations

- ▶ orthogonality, realness, symmetry, finite support (Haar)

Approaches used for simple designs (& more involved as well)

- ▶ relaxing properties: IIR, biorthogonal, complex
- ▶ M -adic MRAs with M integer > 2 or $M = p/q$
- ▶ alternative tilings, less isotropic decompositions
- ▶ with pyramidal-scheme: steerable Marr-like pyramids
- ▶ relaxing critical sampling with oversampled filter banks
- ▶ complexity: (fractional/directional) **Hilbert**, Riesz, phaselets, monogenic, hypercomplex, quaternions, Clifford algebras

Oriented, \pm separable

Illustration of a combination of Hilbert pairs and M -band MRA

$$\widehat{\mathcal{H}\{f\}}(\omega) = -i \operatorname{sign}(\omega) \hat{f}(\omega)$$

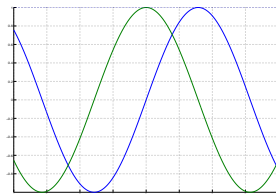


Figure: Hilbert pair 1

Illustration of a combination of Hilbert pairs and M -band MRA

$$\widehat{\mathcal{H}\{f\}}(\omega) = -i \operatorname{sign}(\omega) \widehat{f}(\omega)$$

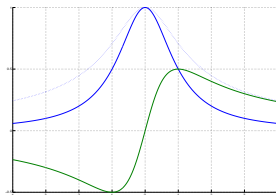


Figure: Hilbert pair 2

Oriented, \pm separable

Illustration of a combination of Hilbert pairs and M -band MRA

$$\widehat{\mathcal{H}\{f\}}(\omega) = -i \operatorname{sign}(\omega) \widehat{f}(\omega)$$

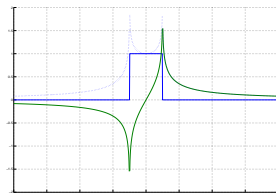


Figure: Hilbert pair 3

Oriented, \pm separable

Illustration of a combination of Hilbert pairs and M -band MRA

$$\widehat{\mathcal{H}\{f\}}(\omega) = -i \operatorname{sign}(\omega) \widehat{f}(\omega)$$

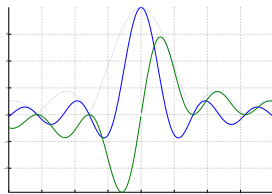


Figure: Hilbert pair 4

$$\widehat{\mathcal{H}\{f\}}(\omega) = -i \operatorname{sign}(\omega) \widehat{f}(\omega)$$

Compute two wavelet trees in parallel, wavelets forming Hilbert pairs, and combine, either with standard 2-band or 4-band

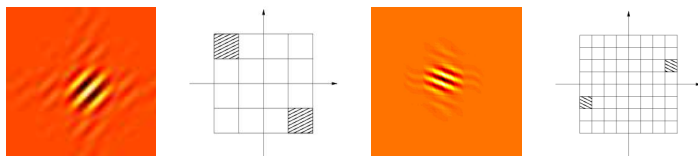


Figure: Examples of atoms and associated frequency partitioning



Figure: GT (reminder)

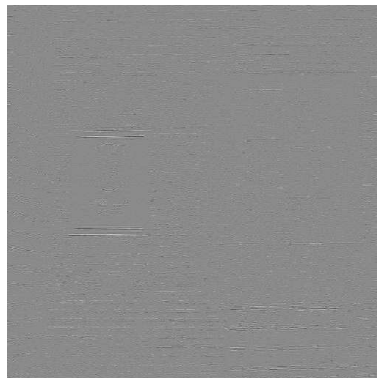


Figure: GT for horizontal subband(s): 2-band, real-valued wavelet

Oriented, \pm separable

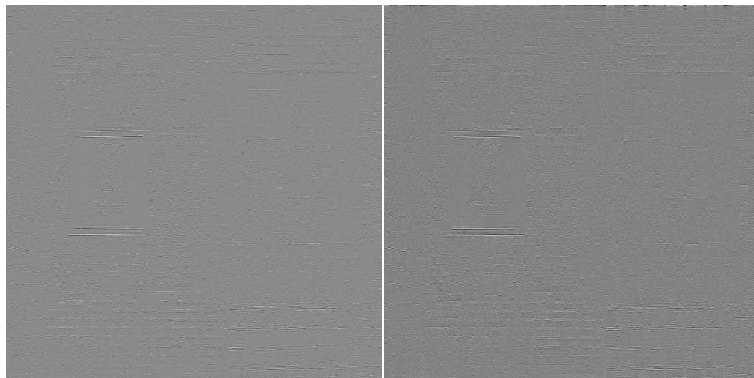


Figure: GT for horizontal subband(s): 2-band dual-tree wavelet

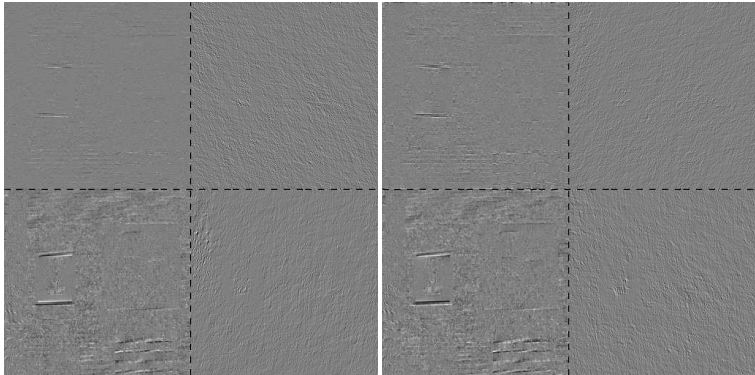


Figure: GT for horizontal subband(s): 4-band dual-tree wavelet

Directional, non-separable

Non-separable decomposition schemes, directly n -D

- ▶ non-diagonal subsampling operators & windows
- ▶ non-rectangular lattices (quincunx, skewed)
- ▶ use of lifting scheme
- ▶ non-MRA directional filter banks
- ▶ steerable pyramids
- ▶ M -band non-redundant directional discrete wavelets
- ▶ building blocks for
 - ▶ contourlets, surfacelets
 - ▶ first generation curvelets with (pseudo-)polar FFT, loglets, directionlets, digital ridgelets, tetrolets

Directional, non-separable

Directional wavelets and frames with actions of rotation or similitude groups

$$\psi_{(\mathbf{b}, a, \theta)}(\mathbf{x}) = \frac{1}{a} \psi\left(\frac{1}{a} R_{\theta}^{-1}(\mathbf{x} - \mathbf{b})\right),$$

where R_{θ} stands for the 2×2 rotation matrix

$$W_f(\mathbf{b}, a, \theta) = \langle \psi_{(\mathbf{b}, a, \theta)}, f \rangle$$

inverted through

$$f(\mathbf{x}) = c_{\psi}^{-1} \int_0^{\infty} \frac{da}{a^3} \int_0^{2\pi} d\theta \int_{\mathbb{R}^2} d^2\mathbf{b} \quad W_f(\mathbf{b}, a, \theta) \psi_{(\mathbf{b}, a, \theta)}(\mathbf{x})$$

Directional, non-separable

Directional wavelets and frames:

- ▶ possibility to decompose and reconstruct an image from a discretized set of parameters; often isotropic
- ▶ examples: Conic-Cauchy wavelet, Morlet/Gabor frames

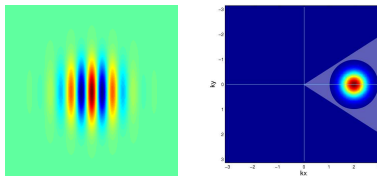


Figure: Morle Wavelet (real part) and Fourier representation

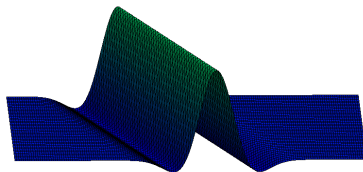
$$\mathcal{R}_f(b, a, \theta) = \int \psi_{(\mathbf{b}, a, \theta)}(\mathbf{x}) f(\mathbf{x}) \mathrm{d}^2 \mathbf{x} = \int \Re_f(\theta, t) a^{-1/2} \psi((t-b)/a) \mathrm{d} t$$


Figure: Ridgelet atom and GT decomposition

Directional, anisotropic scaling

Curvelet transform: continuous and frame

- curvelet atom: scale s , orient. $\theta \in [0, \pi)$, pos. $\mathbf{y} \in [0, 1]^2$:

$$\psi_{s,\mathbf{y},\theta}(\mathbf{x}) = \psi_s(R_\theta^{-1}(\mathbf{x} - \mathbf{y}))$$

$\psi_s(\mathbf{x}) \approx s^{-3/4} \psi(s^{-1/2}x_1, s^{-1}x_2)$ parabolic stretch; ($w \simeq \sqrt{l}$)
 C^2 in C^2 : $O(n^{-2} \log^3 n)$

- tight frame: $\psi_{\mathbf{m}}(\mathbf{x}) = \psi_{2^j, \theta_\ell, \mathbf{x}_n}(\mathbf{x})$ where $\mathbf{m} = (j, n, \ell)$ with sampling locations:

$$\theta_\ell = \ell \pi 2^{\lfloor j/2 \rfloor - 1} \in [0, \pi) \quad \text{and} \quad \mathbf{x}_n = R_{\theta_\ell}(2^{j/2}n_1, 2^j n_2) \in [0, 1]^2$$

- related transforms: shearlets, type-I ripplets

Figure: A curvelet atom and the wedge-like frequency support

Directional, anisotropic scaling

Curvelet transform: continuous and frame

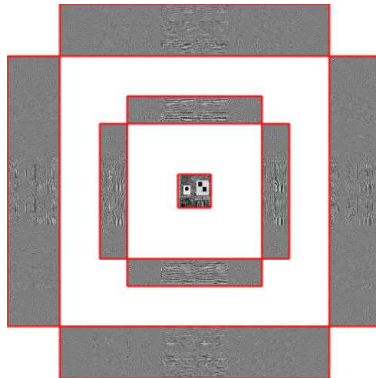


Figure: GT curvelet decomposition

Contourlets: Laplacian pyramid + directional FB

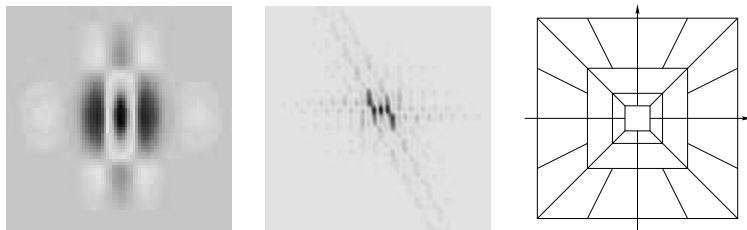


Figure: Contourlet atom and frequency tiling

Directional, anisotropic scaling

Contourlets: Laplacian pyramid + directional FB

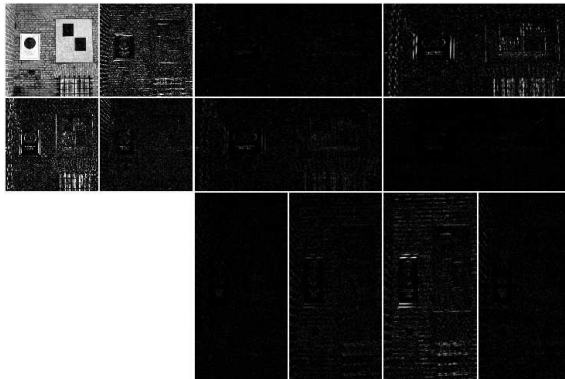


Figure: Contourlet GT (flexible) decomposition

Directional, anisotropic scaling

Additional transforms

- ▶ previously mentioned transforms are better suited for edge representation
- ▶ oscillating textures may require more appropriate transforms
- ▶ examples:
 - ▶ wavelet and local cosine packets
 - ▶ best packets in Gabor frames
 - ▶ brushlets [Meyer, 1997; Borup, 2005]
 - ▶ wave atoms [Demanet, 2007]

Lifting representations

Lifting scheme is an unifying framework

- ▶ to design adaptive biorthogonal wavelets
- ▶ use of spatially varying local interpolations
- ▶ at each scale j , a_{j-1} are split into a_j^o and d_j^o
- ▶ wavelet coefficients d_j and coarse scale coefficients a_j : apply (linear) operators $P_j^{\lambda_j}$ and $U_j^{\lambda_j}$ parameterized by λ_j

$$d_j = d_j^o - P_j^{\lambda_j} a_j^o \quad \text{and} \quad a_j = a_j^o + U_j^{\lambda_j} d_j$$

It also

- ▶ guarantees perfect reconstruction for arbitrary filters
- ▶ adapts to non-linear filters, morphological operations
- ▶ can be used on non-translation invariant grids to build wavelets on surfaces

Lifting representations

$$d_j = d_j^o - P_j^{\lambda_j} a_j^o \quad \text{and} \quad a_j = a_j^o + U_j^{\lambda_j} d_j$$

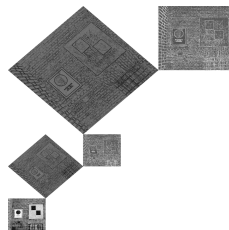
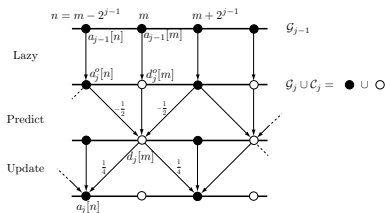


Figure: Predict and update lifting steps and MaxMin lifting of GT

Lifting representations

Extensions and related works

- ▶ adaptive predictions:
 - ▶ possibility to design the set of parameter $\lambda = \{\lambda_j\}_j$ to adapt the transform to the geometry of the image
 - ▶ λ_j is called an association field, since it links a coefficient of a_j^o to a few neighboring coefficients in d_j^o
 - ▶ each association is optimized to reduce the magnitude of wavelet coefficients d_j , and should thus follow the geometric structures in the image
 - ▶ may shorten wavelet filters near the edges
- ▶ grouplets: association fields combined to maintain orthogonality



- ▶ *If you only have a hammer, every problem looks like a nail*
 - ▶ Maslow law of instrument can not be used anymore
- ▶ *The map is not the territory*: incomplete panorama
- ▶ from mild hybridization to GMO-let?
 - ▶ awaited progresses on asymptotics, optimization, models
 - ▶ l_0 not practical, toward structured sparsity
 - ▶ learned frames, robust low-rank approximations, l_0/l_1 -PCA





Conclusion: on a panorama



Acknowledgments:

- ▶ C. Vachier; J. Angulo, M. Moreaud for recent discussions
- ▶ L. Jacques, C. Chaux, G. Peyré
- ▶ to the many *-lets, esp. the forgotten ones

Conclusion: on a panorama



Postponed references & toolboxes & links

- A Panorama on Multiscale Geometric Representations, Intertwining Spatial, Directional and Frequency Selectivity, Signal Processing, Dec. 2011

<http://www.sciencedirect.com/science/article/pii/S0165168411001356>

<http://www.laurent-duval.eu/siva-wits-where-is-the-starlet.html>

<http://www.laurent-duval.eu/siva-panorama-multiscale-geometric-representations.html>