Ondelettes, représentations bidimensionnelles, multi-échelles et géométriques pour les images

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IFP Énergies nouvelles

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In just one slide

Figure: A standard separable wavelet decomposition

Where do we go from here?
Images are pixels (but...):

- suffices for (simple) data (simple) manipulation
  - counting, enhancement, filtering
- very limited in higher level understanding tasks

**Figure:** Image as a linear combination of pixels

\[
\tilde{X} = \begin{pmatrix}
67 & 93 & 129 & 155 \\
52 & 97 & 161 & 207 \\
33 & 78 & 143 & 188 \\
22 & 48 & 84 & 110
\end{pmatrix}
\]
Images are pixels (but...):

A review in an active research field:

- (partly) inspired by:
  - early vision observations
  - sparse coding: wavelet-like oriented filters and receptive fields of simple cells (visual cortex), [Olshausen et al.]
  - a widespread belief in sparsity

- motivated by image handling (esp. compression)
- continued from the first successes of wavelets
- aimed either at pragmatic or heuristic purposes
  - known formation model or unknown information
- developed through a quantity of *-lets and relatives
Images are pixels, but resolution (scale?) matters

Figure: RRRrrrr: coarse to fine! [Chabat et al., 2004]

- notion of sufficient resolution for understanding
- coarse-to-fine and fine-to-coarse links
- impact on more complex images?
Images are different (but...)

Figure: Different kinds of images
Images are different (but...)

Figure: Different kinds of images

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Images are different, but might be described by models

To name a few:

- **edge cartoon + texture:**

  [Meyer-2001]

  \[
  \inf_u E(u) = \int_{\Omega} |\nabla u| + \lambda \|v\|_*, \, f = u + v
  \]

- **edge cartoon + texture + noise:**

  [Aujol-Chambolle-2005]

  \[
  \inf_{u,v,w} F(u, v, w) = J(u) + J^* \left( \frac{v}{\mu} \right) + B^* \left( \frac{w}{\lambda} \right) + \frac{1}{2\alpha} \|f - u - v - w\|_{L^2}
  \]

- **piecewise-smooth + contours + geometrical textures + unmodeled**
Images are different, but might be described by models

Remember: contours and textures are usually not very well-defined

Figure: Real world image and illusions
Images are different, but might be described by models

Remember: contours and textures are usually not very well-defined

important influence of the context

- a variety of methods for description/detection/modeling: smooth curve or polynomial fit, oriented regularized derivatives, discrete geometry, parametric curve detectors (such as the Hough transform), mathematical morphology, EMD, local *frequency estimators*, optical flow approaches, smoothed random models, etc.
Images are different, but might be described by models

Efficient descriptions needed...

- for: compression, denoising, enhancement, inpainting, restoration, fusion, super-resolution, registration, segmentation, reconstruction, source separation, image decomposition, multiple description coding, sparse sampling, etc.
- in a word: understanding (for dedicated purposes)
- first: a brief review of early approaches: from orthogonality (Fourier) to redundancy, and back (DWT)
Images are different, but might be described by models

Efficient descriptions needed... using an assumption of sparsity

Figure: Another real world image with genuine contours and textures and its singular values
Images are different: a guiding thread

Figure: Memorial plaque in honor of A. Haar and F. Riesz: A szegedi matematikai iskola világhírű megalapítói, court. Prof. K. Szatmáry
Guiding thread (GT): early days

Fourier approach: critical, orthogonal

Figure: GT luminance component amplitude spectrum (log-scale)

Fast, compact, very practical but not quite informative
Guiding thread (GT): early days

Scale-space approach: (highly)-redundant, more local

Figure: GT with Gaussian scale-space decomposition

Gaussian filters and heat diffusion interpretation
Varying persistence of features across scales
Guiding thread (GT): early days

Scale-space approach: (less)-redundant, more local

Figure: GT with Gaussian scale-space decomposition

Gaussian filters and heat diffusion interpretation
Varying persistence of features across scales + subsampling
Guiding thread (GT): early days

Differences in scale-space with subsampling

Figure: GT with Laplacian pyramid decomposition

Laplacian pyramid: complete, reduced redundancy, enhances image singularities, low activity regions with small coefficients
Guiding thread (GT): early days

Isotropic wavelets

Consider

Wavelet \( \psi \in L^2(\mathbb{R}^2) \) such that \( \psi(x) = \psi_{\text{rad}}(\|x\|) \), with \( x = (x_1, x_2) \), for some radial function \( \psi_{\text{rad}} : \mathbb{R}^+ \to \mathbb{R} \) (with adm. conditions).

Decomposition and reconstruction

For \( \psi_{(b,a)}(x) = \frac{1}{a} \psi(\frac{x-b}{a}) \), \( W_f(b,a) = \langle \psi_{(b,a)}, f \rangle \) with reconstruction:

\[
 f(x) = \frac{2\pi}{c_\psi} \int_0^{+\infty} \int_{\mathbb{R}^2} W_f(b,a) \psi_{(b,a)}(x) \, d^2b \, \frac{da}{a^3} \tag{1}
\]

if \( c_\psi = (2\pi)^2 \int_{\mathbb{R}^2} |\hat{\psi}(k)|^2 / \|k\|^2 \, d^2k < \infty. \)
Guiding thread (GT): early days

Figure: Example: Marr wavelet as a singularity detector

Multiscale edge detector, more potential wavelet shapes (DoG, Cauchy, etc.)
Guiding thread (GT): early days

**Definition**

The family $\mathcal{B}$ is a frame if there exist two constants $0 < \mu_1 \leq \mu_2 < \infty$ such that for all $f$

$$\mu_1 \|f\|^2 \leq \sum_m |\langle \psi_m, f \rangle|^2 \leq \mu_2 \|f\|^2$$

Possibility of discrete orthogonal bases. In 2D:

**Definition**

Separable orthogonal wavelets: dyadic scalings and translations

$$\psi_m(x) = 2^{-j} \psi^k(2^{-j} x - n)$$

of three tensor-product 2-D wavelets

$$\psi^V(x) = \psi(x_1) \varphi(x_2), \quad \psi^H(x) = \varphi(x_1) \psi(x_2), \quad \psi^D(x) = \psi(x_1) \psi(x_2)$$
Guiding thread (GT): early days

DWT, back to orthogonality: fast, compact and informative, but...

Figure: Separable wavelet decomposition of GT

is it sufficient (singularities, noise, shifts, rotations)?

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Agenda

- A quindecennial panorama of improvements (≥1998)
  - sparser representations of contours and textures through increased spatial, directional and frequency selectivity
  - from fixed to adaptive, from low to high redundancy
  - generally fast, compact, informative, practical
  - bits of hybridization and small paths
  - only a “subsampling”, references postponed to the end

- Outline
  - oriented & geometrical: outcrops, non-separable directionality, anisotropic scaling
  - redundant & adaptive: pursuits, trees and lifting
  - non-Euclidian geometries
  - conclusions
  - local applications
Outcrops from 1-D separable representations

To tackle orthogonal DWT limitations

- orthogonality, realness, symmetry, finite support (Haar)

Approaches used for simple designs (& more involved as well)

- relaxing properties: IIR, biorthogonal, complex
- $M$-adic MRAs with $M$ integer $> 2$ or $M = p/q$
- alternative tilings, less isotropic decompositions
- with pyramidal-scheme: steerable Marr-like pyramids
- relaxing critical sampling with oversampled filter banks
- complexity: (fractional/directional) Hilbert, (Riesz), phaselets, monogenic, hypercomplex, quaternions, Clifford algebras
Outcrops from 1-D separable representations

Illustration of a combination of Hilbert pairs and $M$-band MRA

$$\mathcal{H}\{f\}(\omega) = -\iota \text{sign}(\omega) \hat{f}(\omega)$$

Compute two wavelet trees in parallel, wavelets forming Hilbert pairs, and combine, either with standard 2-band or 4-band

**Figure:** Examples of atoms and associated frequency partitioning
Outcrops from 1-D separable representations

Figure: GT for horizontal subband(s): dyadic, 2-band and 4-band DTT

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Non-separable directionality

Non-separable decomposition schemes, directly $n$-D

- non-diagonal subsampling operators & windows
- non-rectangular lattices (quincunx, skewed)
- use of lifting scheme
- non-MRA directional filter banks
- steerable pyramids
- $M$-band non-redundant directional discrete wavelets
- building blocks for
  - contourlets, surfacelets
  - first generation curvelets with (pseudo-)polar FFT, loglets, directionlets, digital ridgelets, tetrolets
Non-separable directionality

Directional wavelets and frames with actions of rotation or similitude groups

\[ \psi(b,a,\theta)(x) = \frac{1}{a} \psi\left(\frac{1}{a} R_\theta^{-1}(x - b)\right), \]

where \( R_\theta \) stands for the \( 2 \times 2 \) rotation matrix

\[ W_f(b, a, \theta) = \langle \psi(b,a,\theta), f \rangle \]

inverted through

\[ f(x) = c_\psi^{-1} \int_0^\infty \frac{da}{a^3} \int_0^{2\pi} d\theta \int_{\mathbb{R}^2} d^2b \quad W_f(b, a, \theta) \quad \psi(b,a,\theta)(x) \]
Non-separable directionality

Directional wavelets and frames:
- possibility to decompose and reconstruct an image from a discretized set of parameters
- examples: Conic-Cauchy wavelet, Morlet/Gabor frames

Figure: Morlet Wavelet (real part) and Fourier representation
Anisotropic scaling

Ridgelets: 1-D wavelet and Radon transform $R_f(\theta, t)$

$$R_f(b, a, \theta) = \int \psi_{(b,a,\theta)}(x) f(x) \, d^2x = \int R_f(\theta, t) \, a^{-1/2} \psi((t-b)/a) \, dt$$

Figure: Ridgelet atom and GT decomposition
Anisotropic scaling

Curvelet transform: continuous and frame

- curvelet atom: scale $s$, orient. $\theta \in [0, \pi)$, pos. $y \in [0, 1]^2$:

$$\psi_{s,y,\theta}(x) = \psi_s(R_\theta^{-1}(x - y))$$

$$\psi_s(x) \approx s^{-3/4} \psi(s^{-1/2}x_1, s^{-1}x_2) \text{ parabolic stretch; } (w \approx \sqrt{l})$$

$C^2$ in $C^2$: $O(n^{-2} \log^3 n)$

- tight frame: $\psi_m(x) = \psi_{2j, \theta_\ell, x_n}(x)$ where $m = (j, n, \ell)$ with sampling locations:

$$\theta_\ell = \ell \pi 2^{[j/2] - 1} \in [0, \pi) \text{ and } x_n = R_{\theta_\ell}(2^{j/2}n_1, 2^j n_2) \in [0, 1]^2$$

- related transforms: shearlets, type-I ripplets
Anisotropic scaling

Curvelet transform: continuous and frame

![Curvelet atom and the wedge-like frequency support](image)

**Figure**: A curvelet atom and the wedge-like frequency support
Anisotropic scaling

Curvelet transform: continuous and frame

Figure: GT curvelet decomposition
Anisotropic scaling

Contourlets: Laplacian pyramid + directional FB

Figure: Contourlet atom and frequency tiling
Anisotropic scaling

Contourlets: Laplacian pyramid + directional FB

Figure: Contourlet GT (flexible) decomposition
Anisotropic scaling

Additional transforms

- previously mentioned transforms are better suited for edge representation
- oscillating textures may require more appropriate transforms
- examples:
  - wavelet and local cosine packets
  - best packets in Gabor frames
  - brushlets [Meyer, 1997; Borup, 2005]
  - wave atoms [Demanet, 2007]
Pursuit in redundant dictionaries

Highly redundant representations improve the representation of complicated images (with edges and textures):

Approx. $f_M$ of $f$ with $M$ atoms from $\mathcal{B} = \{\psi_{m_j} : 1 \leq j \leq P\}$:

$$f_M = \Psi a = \sum_j a_j \psi_{m_j}, \quad \text{with} \quad \|a\|_0 = \# \{j : a_j \neq 0\} \leq M.$$ 

is generally NP-hard.

- matching pursuit: greedy
- basis pursuit: sparse approximation with

$$f_M = \Psi a = \sum_j a_j \psi_{m_j}, \quad a \in \text{argmin} \frac{1}{2} \|f - \sum_j \tilde{a}_j \psi_{m_j}\|^2 + \mu \|\tilde{a}\|_1,$$
Pursuit in redundant dictionaries

Parametric dictionaries are obtained from basic operations (like rotation, translation, dilation, shearing, modulation, etc.) applied to a continuous mother function.

▶ pursuits in parametric dictionaries: given a set of $S$ transformations $T^i_{m_i}$ for $1 \leq i \leq S$ parameterized by $m_i \in \Lambda_i \subset \mathbb{R}^{n_i}$, the parametric dictionary is related to a certain discretization of $\Lambda^d \subset \Lambda = \Lambda_1 \times \cdots \times \Lambda_S$, i.e.

$$\mathcal{B} = \{ \psi_m(x) = [T^1_{m_1} \cdots T^S_{m_S} \psi](x) \in L^2(\mathbb{R}^2) \}$$

$n = (m_1, \cdots, m_S) \in \Lambda^d$

▶ dictionary discretization may be refined during MP iterations
Tree-structured best basis representations

Figure: Dyadic subdivision of $[0, 1]^2$ in squares $S_{j,i}$ and corresponding quad-tree $\lambda$

- quadtree-based dictionaries
- best basis selection
- adaptive approximation
  - wedgelets, platelets, bandlets
  - adaptive approximations over the wavelet domain: wedgeprints, 2nd gen. bandlets, etc.
Lifting representations

Lifting scheme is an unifying framework

- to design adaptive biorthogonal wavelets
- use of spatially varying local interpolations
- at each scale $j$, $a_{j-1}$ are split into $a_j^o$ and $d_j^o$
- wavelet coefficients $d_j$ and coarse scale coefficients $a_j$: apply (linear) operators $P_{j}^{\lambda_j}$ and $U_{j}^{\lambda_j}$ parameterized by $\lambda_j$

$$d_j = d_j^o - P_{j}^{\lambda_j} a_j^o \quad \text{and} \quad a_j = a_j^o + U_{j}^{\lambda_j} d_j$$

It also

- guarantees perfect reconstruction for arbitrary filters
- adapts to non-linear filters, morphological operations
- can be used on non-translation invariant grids to build wavelets on surfaces
Lifting representations

\[
d_j = d_j^o - P_{j}^{\lambda_j}a_j^o \quad \text{and} \quad a_j = a_j^o + U_{j}^{\lambda_j}d_j
\]

Figure: Predict and update lifting steps and MaxMin lifting of GT
Lifting representations

Extensions and related works

- adaptive predictions:
  - possibility to design the set of parameter $\lambda = \{\lambda_j\}_j$ to adapt the transform to the geometry of the image
  - $\lambda_j$ is called an association field, since it links a coefficient of $a_j^o$ to a few neighboring coefficients in $d_j^o$
  - each association is optimized to reduce the magnitude of wavelet coefficients $d_j$, and should thus follow the geometric structures in the image
  - may shorten wavelet filters near the edges

- grouplets: association fields combined to maintain orthogonality
Images are not (all) flat

On the sphere, motivated by astronomy and geosciences

- filtering (two-angle spherical param. \( \mathbf{x} = (\theta, \varphi) \in S^2 \))

\[
(f \star h)(\rho) = \int_{S^2} h(\rho \mathbf{x}) f(\mathbf{x}) \ d\mu(\mathbf{x}),
\]

where \( \rho \in SO(3) \) is a rotation (driven by three angles) applied to the point \( \mathbf{x} \in S^2 \) and \( d\mu(\mathbf{x}) = \sin \theta d\theta d\varphi \).

- Fourier transform

\[
\hat{f}_{\ell m} = \langle Y_{\ell m}, f \rangle = \int_{S^2} Y_{\ell m}^*(\mathbf{x}) f(\mathbf{x}) \ d\mu(\mathbf{x}), \quad f(\mathbf{x}) = \sum_{\ell, m} \hat{f}_{\ell m} Y_{\ell m}(\mathbf{x})
\]

with respect to orthonormal basis of spherical harmonics

\( \mathcal{Y} = \{ Y_{\ell m}(\mathbf{x}) : \ell \geq 0, |m| \leq \ell \} \), i.e. the eigenvectors of the spherical Laplacian.
Images are not (all) flat

- **spherical Scale-Space:** solution at at time $\tau > 0$ initialized to $f$
  \[
  f(\xi, \tau) = \sum_{\ell, m} \hat{f}_{\ell m}(\tau) Y_{\ell m}(\xi)
  \]
  with $\hat{f}_{\ell m}(\tau) = \hat{f}_{\ell m} e^{-\ell(\ell+1)\tau}$ and $f(\xi, 0) = f(\xi)$

- **spectral Wavelets**
  - spherical wavelets $\psi_a(\xi)$, $W_f(a, \xi) = (f * \psi_a)(\xi)$
  \[
  f(\xi') = \langle f \rangle + \int_{\mathbb{R}^+} \int_{S^2} W_f(a, \xi) \psi_a(\xi' \cdot \xi) \frac{da}{a} d\xi
  \]
  with $\langle f \rangle = \frac{1}{4\pi} \int_{S^2} f(\xi) d\mu(\xi)$

- **MRA on the sphere with QMF**
- **needlets**
Images are not (all) flat

Some of the other (many) constructions

- stereographic wavelets
- Haar transforms (embedded spherical triangulations)

\[ j = 5 \quad j = 4 \quad j = 3 \quad j = 2 \]

**Figure:** Projection on the spherical Haar multiresolution

- steerable wavelets the sphere
- HEALPix: trans. wavelets, ridgelets, curvelets on the sphere
- SOHO wavelets, FLaglets, S2lets
Images are not (all) flat

Some of the other (many) constructions
- wavelets on general 2-manifolds
- lifting scheme on meshed surfaces

Figure: Multiresolution semi-regular mesh
Images are not (all) flat

Some of the other (many) constructions

- wavelets on general 2-manifolds
- lifting scheme on meshed surfaces

Figure: Surface approximation by thresholding (100%, 10%, 5%, 2%)
Images are not images anymore

Figure: Image as a graph

Pixels in images may be viewed as *loci*:

- with connectivity information (even non-local)
- scalar, vector-valued
- labeled
Images are not images anymore

Figure: Wavelets on a swiss roll

Wavelets on graphs [Hammond et al., 2011]
Conclusion: on a panorama

A panorama of 2-D images to a 3-D scene is not seamless:

- *If you only have a hammer, every problem looks like a nail*
  - Maslow law of instrument can not be used anymore
- *The map is not the territory*: incomplete panorama
- from mild hybridization to GMO-let?
  - awaited progresses on asymptotics, optimization, models
  - $l_0$ not practical, toward structured sparsity
  - learned frames, robust low-rank approximations, $l_0/l_1$-PCA
Conclusion: on a panorama

Take-away messages anyway?

- is there a best?
  - even more complex to determine than with (discrete) wavelets
  - intricate relationship: sparsifying transform/associated processing
- wishlist: fast, mild redundancy, some invariance, manageable correlation in transformed domain, fast decay, tunable frequency decomposition, complex or more (phase issues)
Conclusion: on a panorama

Acknowledgments:

- L. Jacques, C. Chaux, G. Peyré and reviewers
- to the many *-lets, especially the forgotten ones

Postponed references & toolboxes & links

- more stuff:
Conclusion: to be continued

Room(let) for improvement:

Activelet, AMlet, Armlet, Bandlet, Barlet, Bathlet, Beamlet, Binlet, Bumplet, Brushlet, Caplet, Camplet, Chirplet, Chordlet, Circlet, Coiflet, Contourlet, Cooklet, Craplet, Cubelet, CURElet, Curvelet, Daublet, Directionlet, Dreamlet, Edgelet, Famlet, FLaglet, Flatlet, Fourierlet, Framelet, Fresnelet, Gaborlet, GAMlet, Gausslet, Graphlet, Grouplet, Haarlet, Haardlet, Heatlet, Hutlet, Hyperbolet, Icalet (Icalette), Interpolet, Loglet, Marrlet, MIMOlet, Monowavelet, Morelet, Morphlet, Multiselectivelet, Multiwavelet, Needlet, Noiselet, Ondelette, Ondulette, Prewavelet, Phaselet, Planelet, Platelet, Purelet, QVlet, Radonlet, RAMlet, Randlet, Ranklet, Ridgelet, Riezlet, Ripplet (original, type-I and II), Scalet, S2let, Seamlet, Seislet, Shadelet, Shapelet, Shearlet, Sincllet, Singlet, Slanlet, Smoothlet, Snakelet, SOHOlet, Sparselet, Spikelet, Splinelet, Starlet, Steerlet, Stockeslet, SURE-let (SURElet), Surfaclelet, Surflet, Symmlet, S2let, Tetrolet, Treelet, Vaguelette, Wavelet-Vaguelette, Wavelet, Warblet, Warplet, Wedgelet, Xlet, what's next?
Local applications

Weak interest in Lenna-like images

Figure: Seismic Lenna
Local applications

Figure: Surface wave removal (before)
Local applications

Figure: Surface wave removal (after)
Local applications

Figure: Paper to seismics

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Figure: Catalyst measurements

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Figure: Catalyst measurements